

# EMBEDDED CONTACT HOMOLOGY AND OPEN BOOK DECOMPOSITIONS

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**ABSTRACT.** This is the first of a series of papers devoted to proving the equivalence of Heegaard Floer homology and embedded contact homology (abbreviated ECH). In this paper we prove that, given a closed, oriented, contact 3-manifold, there is an equivalence between ECH of the closed 3-manifold and a version of ECH, defined on the complement of the binding of an adapted open book decomposition.

## 1. INTRODUCTION

This is the first of a series of papers devoted to proving the equivalence of Heegaard Floer homology, defined by Ozsváth-Szabó [OSz1, OSz2], and embedded contact homology (abbreviated ECH), defined by Hutchings [Hu, Hu2]. An alternate proof of the  $HF = ECH$  correspondence has recently been announced by Kutluhan-Lee-Taubes [KLT].

Let  $M$  be a closed oriented 3-manifold. The Heegaard Floer homology groups  $HF^+(M)$  and  $\widehat{HF}(M)$  are defined in terms of a Heegaard decomposition of  $M$  and were shown by Ozsváth-Szabó [OSz1, OSz2] to be invariants of the manifold  $M$ . On the other hand, the embedded contact homology groups  $ECH(M)$  and  $\widehat{ECH}(M)$  — defined as the mapping cone of a  $U$ -map (see Section 3.6) — are defined using a contact form  $\alpha$  on  $M$  and an adapted almost complex structure  $J$  on the symplectization  $\mathbb{R} \times M$ . There is currently no direct proof of the fact that the ECH groups are invariants of  $M$ ; the only known proof is due to Taubes [T1, T2], and is a consequence of the equivalence between Seiberg-Witten Floer (co-)homology and ECH.

The goal of this paper is to modify the definition of ECH so that it is easier to define chain maps to and from Heegaard Floer homology. (The chain maps will be defined in the sequel [CGH].) This will be done in the

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context of *open book decompositions* and the *Giroux correspondence* [Gi2], which we describe now. Consider the pair  $(S, h)$  consisting of a compact, oriented, connected surface  $S$  with nonempty boundary (sometimes called a *bordered surface*) and a diffeomorphism  $h : S \xrightarrow{\sim} S$  which restricts to the identity on  $\partial S$ . Given a link  $K \subset M$ , we say that  $M$  admits an *open book decomposition* with fiber  $S$ , monodromy  $h$ , and *binding*  $K$  if there is a (locally trivial) fibration  $M \setminus K \rightarrow S^1$  with fiber  $\text{int}(S)$  and monodromy  $h$ . By the work of Giroux [Gi2], there is a one-to-one correspondence between isomorphism classes of contact structures on  $M$  and open book decompositions of  $M$  modulo stabilizations.

There is a related manifold  $N = N_h$  which we call the *suspension of*  $(S, h)$  and which is constructed by taking  $S \times [0, 1]$  with coordinates  $(x, t)$  and identifying  $(x, 1) \sim (h(x), 0)$ . We can define a contact form  $\alpha$  on  $N$  so that the first return map of the Reeb vector field  $R_\alpha$  is isotopic to  $h$  relative to  $\partial S$ . The construction is done in some detail in Section 2.

The goal of this paper is to introduce the ECH groups  $ECH(N, \partial N, \alpha)$  and  $\widehat{ECH}(N, \partial N, \alpha)$ , which are defined only in terms of  $N$  and  $\alpha$ , and to prove the following:

**Theorem 1.0.1.**

- (1)  $ECH(M) \simeq ECH(N, \partial N, \alpha)$ .
- (2)  $\widehat{ECH}(M) \simeq \widehat{ECH}(N, \partial N, \alpha)$ .

In other words, Theorem 1.0.1 allows us to rephrase the ECH groups of  $M$  in terms of ECH groups on the complement of the binding. We remark here that Yau [Y] and Wendl [We, We2] have examined related issues in their work.

The paper is organized as follows: In Section 2 we construct the contact forms  $\alpha_\delta$  on  $M$  and their restrictions  $\alpha$  on  $N$ . Sections 3 and 4 are devoted to a quick review of ECH and some technicalities involving direct limits. In particular, the ECH groups  $ECH(M)$  and  $\widehat{ECH}(M)$  are defined in Section 3. Section 5 reviews some Morse-Bott theory in the context of ECH. In Section 6 we define certain ECH groups for compact manifolds with torus boundary. Then in Section 7 we define the variants  $ECH(N, \partial N, \alpha)$  and  $\widehat{ECH}(N, \partial N, \alpha)$  of ECH which appear in Theorem 1.0.1; we also give an intuitive explanation of why Theorem 1.0.1 is true. In Section 8 we calculate some ECH groups on solid tori which appear in the proof of Theorem 1.0.1. Section 9 then completes the proof of Theorem 1.0.1.

## 2. CONSTRUCTIONS OF CONTACT FORMS

Unless otherwise stated, in this paper all 3-manifolds are oriented and all contact structures are oriented and positive.

**2.1. Open book decompositions.** For the purposes of this paper, we use a constructive definition of an open book decomposition: Given the pair  $(S, h)$ , let  $N_h$  be the suspension  $(S \times [0, 1]) / (x, 1) \sim (h(x), 0)$ , where  $x \in S$  and  $t$  is the coordinate for  $[0, 1]$ . We orient  $\partial S$  as the boundary of  $S$  and choose an oriented identification  $\partial S \simeq \mathbb{R}/\mathbb{Z}$ , where  $\theta$  is the coordinate for  $\mathbb{R}/\mathbb{Z}$ . We then form a slightly enlarged manifold  $N'_h$  by gluing a thickened torus  $T^2 \times [1, 2]$  to  $N_h$ . Here  $T^2 \times \{1\}$  is identified with  $\partial N_h$  in an orientation-preserving manner so that  $T^2 \simeq (\mathbb{R}/\mathbb{Z}) \times [0, 1] / (0 \sim 1)$  has coordinates  $(\theta, t)$  which extend those on  $\partial N_h$  and  $y$  is the coordinate for  $[1, 2]$ . For convenience we write  $T_y = T^2 \times \{y\}$ . Finally we obtain  $M$  by gluing a solid torus  $V$  to  $N'_h$  so that a meridian of  $V$  is mapped to  $\{\theta\} \times ([0, 1] / \sim) \times \{2\}$  for some  $\theta \in \partial S$ , and a longitude of  $V$  is mapped to  $\partial S \times \{t\} \times \{2\}$  for some  $t \in [0, 1]$ . When we want to emphasize the fact that we are using this constructive point of view, we will talk about *abstract open books*.

This construction specifies a natural oriented basis  $\{v_1, v_2\}$  of  $H_1(T_y)$ , where  $v_1$  is the class of  $\partial S$  — by this we mean a closed curve on  $T_y$  isotopic to  $\partial_S$  inside  $T^2 \times [1, 2]$  — with the orientation induced by  $S$ , and  $v_2$  is the class of the meridian with the orientation inherited from  $[0, 1]$ . We will use this basis to identify the isotopy classes of simple closed curves on  $T_y$  with rational numbers so that the meridian has slope  $\infty$  and  $\partial S$  has slope 0.

*Remark 2.1.1.* The above slope convention is the same as the usual surgery convention for performing surgery along the binding.

**2.2. Construction of the contact form on  $N_h$ .** The construction of the contact form(s) on  $M$  adapted to the abstract open book  $(S, h)$  will be done in three steps, corresponding to the three pieces  $N_h$ ,  $T^2 \times [1, 2]$ , and  $V$ .

A 1-form  $\beta$  on  $S$  is a *Liouville 1-form* if  $\omega = d\beta > 0$  and the Liouville vector field  $X$  given by  $i_X d\beta = \beta$  is positively transverse to  $\partial S$ . We take a Liouville form  $\beta$  on  $S$  so that  $\beta = cyd\theta$  in a neighborhood  $N(\partial S) \subset S$  of  $\partial S$ . Here  $c > 0$  is a small constant and  $N(\partial S)$  is identified with  $[1 - \delta, 1] \times \mathbb{R}/\mathbb{Z}$  with coordinates  $(y, \theta)$  which extend those on  $T^2 \times [1, 2]$ .

Although the difference is slight, in this subsection we assume that the diffeomorphism  $h : S \xrightarrow{\sim} S$  satisfies  $h|_{N(\partial S)} = id$ . Let  $\text{Symp}(S, \partial S, \omega)$  be the group of symplectomorphisms of  $(S, \omega)$  which restrict to the identity on a neighborhood of  $\partial S$ . By Moser's lemma, there is an isotopy of  $h$  relative to  $\partial S$  so that the resulting diffeomorphism — also called  $h$  by abuse of notation — is in  $\text{Symp}(S, \partial S, \omega)$ .

**Lemma 2.2.1 (Giroux).** *Given  $h \in \text{Symp}(S, \partial S, \omega)$ , there exists an isotopy  $h_t$ ,  $t \in [0, 1]$ , in  $\text{Symp}(S, \partial S, \omega)$  so that  $h_0 = h$  and  $h_1^* \beta - \beta = df$  for some function  $f$  on  $S$ .*

*Proof.* Let  $\mu = h^*\beta - \beta$  and let  $Y$  be the vector field which satisfies  $i_Y\omega = -\mu$ . By Cartan's formula, we compute that  $\mathcal{L}_Y\omega = i_Yd\omega + d(i_Y\omega) = -d\mu = 0$  and  $\mathcal{L}_Y\mu = i_Yd\mu + d(i_Y\mu) = 0$ . Hence the flow  $\phi_t$  of  $Y$  preserves  $\omega$  and  $\mu$ . Moreover,  $\phi_t$  is equal to the identity near  $\partial S$ , where we have  $\mu = 0$ .

Now let  $h_t = h \circ \phi_t$ . We then compute that:

$$\begin{aligned} \frac{d}{dt}h_t^*\beta &= \phi_t^*(\mathcal{L}_Y h^*\beta) = d(\phi_t^*(i_Y h^*\beta)) + \phi_t^*(i_Y d(h^*\beta)) \\ &= dg_t + \phi_t^*(i_Y\omega) = dg_t - \phi_t^*\mu = dg_t - \mu, \end{aligned}$$

where  $g_t = \phi_t^*(i_Y h^*\beta)$ . Hence

$$(2.2.1) \quad \frac{d}{dt}h_t^*\beta = dg_t + \beta - h^*\beta.$$

By integrating Equation (2.2.1), we obtain  $h_1^*\beta - \beta = df$ , where  $f = \int_0^1 g_t dt$ .  $\square$

Next we construct a contact form on  $N_h$  whose corresponding Reeb vector field is transverse to the fibers and has first return map  $h$ .

**Lemma 2.2.2.** *Let  $h$  be a diffeomorphism in  $\text{Symp}(S, \partial S, \omega)$  which satisfies  $h^*\beta - \beta = df$  for some function  $f$  on  $S$ . Then there is a contact form  $\alpha = f_t dt + \beta_t$  on  $N_h$ , where  $f_t$  is a positive function on  $S$  and  $\beta_t$  is a Liouville 1-form on  $S$ , so that the corresponding Reeb vector field  $R_\alpha$  is transverse to all the fibers  $S \times \{t\}$  and  $h$  is the first return map of  $R_\alpha$ .*

For a more complete discussion of the realizability of surface symplectomorphisms as the first return map of a Reeb vector field, we refer the reader to [CHL].

*Proof.* Consider the contact 1-form  $\alpha = f_t dt + \beta_t$  on  $S \times [0, 1]$ , where  $f_t$  is to be determined,  $\beta_0 = \beta$ ,  $\beta_1 = h^*\beta$ , and

$$\beta_t = \chi(t)\beta_1 + (1 - \chi(t))\beta_0$$

interpolates between  $\beta_0$  and  $\beta_1$ . Here we take  $\chi : [0, 1] \rightarrow [0, 1]$ , so that  $\chi(0) = 0$ ,  $\chi(1) = 1$ ,  $\frac{d\chi}{dt}(t) = \dot{\chi}(t) \geq 0$ , and  $\chi$  is constant near 0 and 1.

Using the condition  $h^*\beta - \beta = d_S f$ , we compute that the 1-form  $\dot{\beta}_t$  is exact on  $S$ :

$$\dot{\beta}_t = \dot{\chi}(t)(\beta_1 - \beta_0) = \dot{\chi}(t)(d_S f) = d_S(\dot{\chi}(t)f).$$

Here  $d_S$  is the exterior derivative in the  $S$ -direction. We then take  $f_t$  to be  $\dot{\chi}(t)f$  plus a sufficiently large positive constant, so that  $f_t > 0$ . Then  $\dot{\beta}_t = d_S f_t$ . Since  $\chi$  is constant near  $t = 0$  and  $t = 1$ ,  $f_t$  is also constant, and so is  $\beta_t$ . In particular, we have  $h^*f_1 = f_0$ .

We now compute that

$$d\alpha = d_S f_t \wedge dt + d_S \beta_t + dt \wedge \dot{\beta}_t = d_S f_t \wedge dt + \omega + dt \wedge d_S f_t = \omega.$$

Hence the Reeb vector field of  $\alpha$  is parallel to  $\partial_t$  on  $S \times [0, 1]$  and its first return map is exactly  $h$ .  $\square$

### 2.3. Construction of the contact form on $T^2 \times [1, 2]$ .

2.3.1. We first analyze contact forms on  $T^2 \times [1, 2]$  with coordinates  $(\theta, t, y)$  which can be written as:

$$(2.3.1) \quad \alpha_{f,g} = g(y)d\theta + f(y)dt.$$

Here  $f, g$  are functions on  $[1, 2]$ ; in particular  $f(y)$  is unrelated to the function  $f$  from the previous subsection.

The following is a straightforward calculation:

#### Lemma 2.3.1.

(i) *The form  $\alpha_{f,g}$  is contact if and only if*

$$(2.3.2) \quad fg' - f'g > 0.$$

(ii) *The kernel  $\xi = \ker \alpha_{f,g}$  is spanned by  $\{\partial_y, -f\partial_\theta + g\partial_t\}$ .*

(iii) *Assuming  $\alpha_{f,g}$  is a contact form, the Reeb vector field  $R = R_{\alpha_{f,g}}$  is:*

$$(2.3.3) \quad R = \frac{1}{fg' - f'g}(-f'\partial_\theta + g'\partial_t),$$

where  $f'$  refers to  $\frac{df}{dy}$  and  $g'$  refers to  $\frac{dg}{dy}$ .

In words, Equation (2.3.2) says that  $(f', g')$  is transverse to the radial rays in the  $fg$ -plane and rotates in the counterclockwise direction.

2.3.2. *Slight modification of  $\alpha$  on  $N_h$ .* From now on we write  $N = N_h$  and  $N' = N'_h$ . The goal of this subsection is to make a slight modification of  $\alpha$  on  $N_h$  so that  $\partial N_h$  becomes a *negative* Morse-Bott family — one that behaves like a sink. (See Section 5.1 for more details.)

Let  $\alpha$  be the contact form on  $N$  which was constructed in Section 2.2. On  $T_1 = \partial N$ , the germ of  $\alpha$  is given by  $f(y)dt + g(y)d\theta$ , where  $f(y) = C$  and  $g(y) = cy$ . Here  $c > 0$  is a small constant and  $C > 0$  is a large constant. We extend  $\alpha$  to  $T^2 \times [1, 1 + \varepsilon]$  by extending  $(f(y), g(y))$  to  $y \in [1, 1 + \varepsilon]$  as follows:

- (1)  $(f(y), g(y))$  satisfies Equation (2.3.2).
- (2)  $(f(y), g(y)), y \in [1, 1 + \varepsilon]$ , is close to  $(f(1), g(1))$ .
- (3)  $(f(y), g(y)) = (f(1 + \varepsilon) + (y - (1 + \varepsilon))^2, g(1 + \varepsilon) + (y - (1 + \varepsilon)))$  near  $y = 1 + \varepsilon$ .

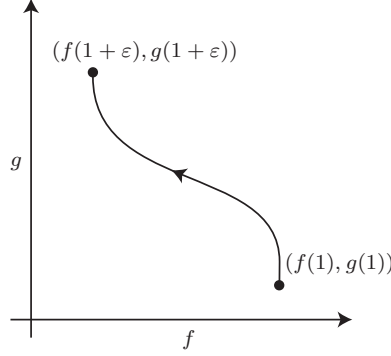


FIGURE 1. Trajectory of  $(f(y), g(y))$ . The  $f$ -axis and  $g$ -axis do not necessarily intersect at  $(0, 0)$  in this figure.

See Figure 1. In particular, Condition (3) implies that  $(f'(1 + \varepsilon), g'(1 + \varepsilon))$  is parallel to  $(0, 1)$ . Hence  $T_{1+\varepsilon}$  is foliated by a Morse-Bott family of Reeb orbits of slope  $\infty$ . We write  $\alpha$  for the extension of  $\alpha$  to  $N \cup (T^2 \times [1, 1 + \varepsilon])$ .

We now consider the deformation retract

$$\phi : N \cup (T^2 \times [1, 1 + \varepsilon]) \xrightarrow{\sim} N,$$

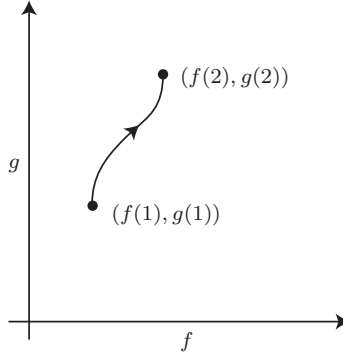
obtained by flowing along the vector field  $X = -a(y)\partial_y$ , where  $a(y) = 1$  on  $T^2 \times [1, 1 + \varepsilon]$  and damps out to zero on  $T^2 \times [1 - \varepsilon, 1]$ . Finally, we perturb  $\phi_*\alpha$  on  $N$  so that all Reeb orbits in  $\text{int}(N)$  become nondegenerate, while keeping  $\partial N$  Morse-Bott. *The resulting form will be called  $\alpha$  in the rest of the paper.*

**2.3.3. Extension to  $N'$ .** We now extend the contact form  $\alpha$  defined on  $N$  to  $N' = N \cup (T^2 \times [1, 2])$ . As before, we extend the pair of functions  $(f, g) = (f_\delta, g_\delta)$  — here our functions depend on a small positive irrational parameter  $\delta$  — so that the following hold:

- (1)  $(f, g)$  satisfies Equation (2.3.2).
- (2)  $(f(y), g(y))$ ,  $y \in [1, 2]$ , is close to  $(f(1), g(1))$ .
- (3)  $0 \leq \frac{f'(y)}{g'(y)} \leq \delta$ ;  $\frac{f'(y)}{g'(y)}$  is increasing on  $(1, \frac{3}{2})$  and is decreasing on  $(\frac{3}{2}, 2)$ , and is equal to  $\delta$  at  $y = \frac{3}{2}$ .
- (4)  $(f(y), g(y)) = (f(1) + (y - 1)^2, g(1) + (y - 1))$  near  $y = 1$ .
- (5)  $(f(y), g(y)) = (f(2) - (y - 2)^2, g(2) + (y - 2))$  near  $y = 2$ .
- (6) For all  $\delta$ ,  $(f_\delta(2), g_\delta(2))$  lie on the same line through the origin.

Refer to Figure 2.

The resulting contact form on  $N'$  will be called  $\alpha_\delta$ . Its Reeb vector field  $R_{\alpha_\delta}$  has Morse-Bott tori whose Reeb orbits have rational slope in the interval  $[-\infty, -\frac{1}{\delta}]$ ; each rational slope occurs twice, once in the interval  $[1, \frac{3}{2}]$


 FIGURE 2. Trajectory of  $(f(y), g(y))$ .

and once in the interval  $[\frac{3}{2}, 2]$ . Note that the Reeb orbits in the two Morse-Bott tori of infinite slope have parallel directions and are in “elimination position”. Also, by taking  $\delta$  to be sufficiently small, all the Reeb orbits in  $\text{int}(T^2 \times [1, 2])$  can be made to have arbitrarily large action and intersect the page an arbitrarily large number of times.

#### 2.4. Construction of the contact form on $V$ .

2.4.1. On the solid torus  $V = D^2 \times S^1$  we choose cylindrical coordinates  $(\rho, \phi, \theta)$  so that  $\partial V = \{\rho = 1\}$ . We identify  $\partial V$  with  $-\partial N'$  so that the point  $(\theta, \phi) \in \partial V$  is identified with  $(\theta, t = \frac{\phi}{2\pi}) \in \partial N'$ ; moreover we set  $\rho = 3 - y$ .

We consider contact forms which can be written as:

$$(2.4.1) \quad \alpha_{f,g} = g(\rho)d\theta + \frac{f(\rho)}{2\pi}d\phi.$$

Here we need to choose  $(f(\rho), g(\rho))$  so that  $\alpha_{f,g}$  is smooth on all of  $S^1 \times D^2$ , which means that  $f(0) = 0$  and the derivatives of odd degree of both  $f$  and  $g$  at  $\rho = 0$  are 0. The analog of Lemma 2.3.1 is the following:

**Lemma 2.4.1.** *The contact condition for  $\alpha_{f,g}$  away from  $\rho = 0$  is:*

$$(2.4.2) \quad f'g - fg' > 0,$$

where  $f' = \frac{df}{d\rho}$  and  $g' = \frac{dg}{d\rho}$ , and the contact condition for  $\alpha_{f,g}$  along  $\rho = 0$  is:

$$\lim_{\rho \rightarrow 0} \frac{f'g - fg'}{\rho} > 0.$$

The contact structure  $\ker \alpha_{f,g}$  is spanned by  $\{\partial_\rho, -f\partial_\theta + 2\pi g\partial_\phi\}$  and the Reeb vector field is given by:

$$(2.4.3) \quad R_{\alpha_{f,g}} = \frac{1}{f'g - fg'}(f'\partial_\theta - 2\pi g'\partial_\phi).$$



In particular,  $R_{\alpha_{f,g}}$  is parallel to  $\partial_\theta$  at  $\rho = 0$ .

Note that Equations (2.3.2) and (2.4.2) have opposite signs, which is due to the fact that  $\frac{d\rho}{dy} = -1$ .

**2.4.2. Some constructions.** Since they will be useful later, we present a pair of constructions of contact forms on  $S^1 \times D^2$  of the form given in Equation (2.4.1). Here  $D^2 = \{\rho \leq 1\}$ .

**Example 1.** Given  $\nu > 0$  and  $C > 1$ , let  $(f(\rho), g(\rho)) = (\nu\rho^2, C - \rho^2)$ . This gives a smooth contact form on  $S^1 \times D^2$ . Then the Reeb vector field on  $T_y$ ,  $y = 3 - \rho$ , will have slope  $-\frac{g'}{f'} = \frac{1}{\nu}$ , where  $f', g'$  are derivatives with respect to  $\rho$ . In particular, if  $\nu$  is irrational, then the only simple closed orbit of  $R_{\alpha_{f,g}}$  is the core curve  $\rho = 0$ .

**Example 2.** Given  $C > 1$ , let  $(f(\rho), g(\rho)) = (\rho^2, C - \rho^4)$ , which also gives a smooth contact form on  $S^1 \times D^2$ . Its Reeb vector field will have slope  $-\frac{g'}{f'} = 2\rho^2$  along  $T_y$ ; this means that the slope will vary monotonically from 0 (which is the core curve) to 2 as  $\rho$  goes from 0 to 1.

**2.4.3. The extension of  $\alpha_\delta$  to  $V$ .** Fix  $\delta_0 > 0$  and a contact form  $\alpha_{\delta_0}$  on  $N'$ . We extend  $\alpha_{\delta_0}$  to  $V$  so that the following hold:

- (1)  $(f_{\delta_0}, g_{\delta_0})$  satisfies Equation (2.4.2).
- (2)  $(f_{\delta_0}(\rho), g_{\delta_0}(\rho)) = (\rho^2, C - \rho^2)$  near  $\rho = 0$ , where  $C > 0$  is a large constant.
- (3)  $(f_{\delta_0}(\rho), g_{\delta_0}(\rho)) = (f_{\delta_0}(1) - (\rho - 1)^2, g_{\delta_0}(1) - (\rho - 1))$  near  $\rho = 1$ .  
Here  $(f_{\delta_0}, g_{\delta_0})|_{\rho=1} = (f_{\delta_0}, g_{\delta_0})|_{y=2}$ .
- (4)  $-\frac{g'_{\delta_0}}{f'_{\delta_0}}$  monotonically increases from 1 to  $+\infty$  as  $\rho$  goes from 0 to 1.

Refer to Figure 3.

We now extend  $\alpha_\delta$  to  $V$  for all sufficiently small  $\delta > 0$  by writing  $(f_\delta, g_\delta)$  as a suitable constant multiple of  $(f_{\delta_0}, g_{\delta_0})$ . This is possible because of Condition (6) in the definition of  $\alpha_\delta$ . Observe that the Reeb orbits of  $\alpha_\delta$  and  $\alpha_{\delta_0}$  agree on  $V$ , modulo parametrization.

On each torus  $T_y \subset S^1 \times D^2$ , the Reeb vector field  $R_{\alpha_\delta}$  gives a foliation by Reeb orbits of slope  $s$  in the interval  $(1, \infty]$ , where there is a unique  $y$  for each slope  $s$ .

**Remark 2.4.2.** The boundary components of the three pieces  $N$ ,  $T^2 \times [0, 1]$ , and  $V$  are Morse-Bott tori of slope  $\infty$ . The Morse-Bott torus  $T_1$  can be perturbed into a pair of simple Reeb orbits: an elliptic orbit  $e$  and a hyperbolic



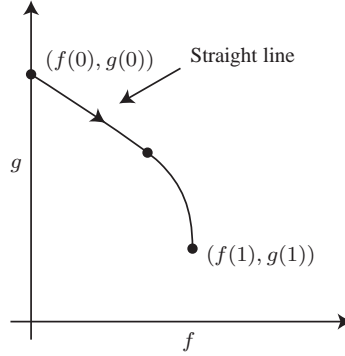


FIGURE 3. Trajectory of  $(f_\delta(\rho), g_\delta(\rho))$ . Here the arrow points in the positive  $\rho$ -direction.

orbit  $h$ . Similarly, the Morse-Bott torus  $T_2$  can be perturbed into a hyperbolic orbit  $h'$  and an elliptic orbit  $e'$ . Their Conley-Zehnder indices with respect to the framing coming from the tori are  $\mu(e) = -1$ ,  $\mu(h) = \mu(h') = 0$ , and  $\mu(e') = 1$ . The verification is left as an exercise.

### 3. REVIEW OF EMBEDDED CONTACT HOMOLOGY

In this section we briefly review the basic definitions of ECH. For more details the reader is referred to [Hu, Hu2]. To avoid orienting the moduli spaces, we will work over  $\mathbb{F} = \mathbb{Z}/2\mathbb{Z}$ .

**3.1. The symplectization.** Let  $M$  be a closed, oriented 3-manifold with a contact form  $\alpha$  so that the Reeb vector field  $R = R_\alpha$  is nondegenerate. Let  $\xi = \ker \alpha$  be the corresponding contact structure. Also choose an almost complex structure  $J$  on  $\mathbb{R} \times M$ , with  $\mathbb{R}$ -coordinate  $s$ , which is *adapted to the symplectization of  $\alpha$*  (or *adapted to  $\alpha$* ), i.e.,

- (1)  $J$  is  $s$ -invariant;
- (2)  $J$  takes  $\xi$  to itself on each  $\{s\} \times Y$ ;
- (3)  $J$  maps  $\partial_s$  to  $R_\alpha$ ;
- (4)  $J|_\xi$  is  $d\alpha$ -positive, i.e.,  $d\alpha(v, Jv) > 0$  for all nonzero  $v \in \xi$ .

**3.2. Generators of the ECH chain complex.** Let  $\mathcal{P}$  be the set of simple orbits of the Reeb vector field  $R_\alpha$ . The ECH chain complex  $ECC(M, \alpha, J)$  is a vector space over  $\mathbb{F}$ , generated by finite multisets  $\gamma = \{(\gamma_i, m_i)\}$ , called *orbit sets*, where:

- $\gamma_i \in \mathcal{P}$  and  $\gamma_i \neq \gamma_j$  for  $i \neq j$ ;
- $m_i$  is a positive integer;
- if  $\gamma_i$  is a hyperbolic orbit, then  $m_i = 1$ .

We will say that  $ECC(M, \alpha, J)$  is *constructed from  $\mathcal{P}$* . An orbit set  $\gamma$  will also be written multiplicatively as  $\prod \gamma_i^{m_i}$ , with the convention that  $\gamma_i^2 = 0$

whenever  $\gamma_i$  is hyperbolic. The empty set  $\emptyset$  will be written multiplicatively as 1.

The homology class of an orbit set  $\gamma$  is

$$[\gamma] = \sum_i m_i [\gamma_i] \in H_1(M).$$

If we want to specify the direct summand generated by orbit sets of class  $A \in H_1(M)$ , we write  $ECC(M, \alpha, J, A)$ .

The *action*  $\mathcal{A}_\alpha(\gamma_i)$  of an orbit  $\gamma_i$  is given by  $\int_{\gamma_i} \alpha$ , and the action of an orbit set  $\gamma$  is given by  $\mathcal{A}_\alpha(\gamma) = \sum_i m_i \mathcal{A}_\alpha(\gamma_i)$ .

**3.3. Moduli spaces.** Let  $\gamma = \{(\gamma_i, m_i)\}$  and  $\gamma' = \{(\gamma'_i, m'_i)\}$  be orbit sets with  $[\gamma] = [\gamma'] \in H_1(M)$ .

The set of finite energy holomorphic maps

$$u : (F, j) \rightarrow (\mathbb{R} \times M, J),$$

modulo holomorphic reparametrizations, which satisfy:

- (1)  $(F, j)$  is a closed Riemann surface with a finite number of punctures removed;
- (2) the neighborhoods of the punctures are mapped asymptotically to cylinders over Reeb orbits;
- (3)  $u(F)$  is asymptotic to  $\mathbb{R} \times \gamma_i$  with total multiplicity  $m_i$  at the positive end of  $\mathbb{R} \times M$ ;
- (4)  $u(F)$  is asymptotic to  $\mathbb{R} \times \gamma'_i$  with total multiplicity  $m'_i$  at the negative end of  $\mathbb{R} \times M$ ;

will be denoted by  $\mathcal{M}_J(\gamma, \gamma')$ . We often refer to an element  $u$  of  $\mathcal{M}_J(\gamma, \gamma')$  as a *holomorphic curve from  $\gamma$  to  $\gamma'$* .

We say that  $J$  is *regular* if, for all orbit sets  $\gamma, \gamma'$  and  $u \in \mathcal{M}_J(\gamma, \gamma')$  which have no multiply-covered components,  $\mathcal{M}_J(\gamma, \gamma')$  is transversely cut out (and hence is a manifold) near  $u$ . Such  $J$  are dense by a result of Dragnev [Dr].

**3.4. The ECH index.** Let  $\gamma = \{(\gamma_i, m_i)\}$  and  $\gamma' = \{(\gamma'_i, m'_i)\}$  be orbit sets and  $Z \in H_2(M, \gamma, \gamma')$ . Pick a trivialization  $\tau$  of  $\xi$  along each orbit in the orbit sets  $\gamma, \gamma'$ . Let  $c_\tau(\xi|_Z)$  be the first Chern class of  $\xi$  evaluated on  $Z$ , relative to the trivialization  $\tau$  on  $\partial Z$ .

The Conley-Zehnder index of an orbit  $\gamma_i$  with respect to  $\tau$  will be written as  $\mu_\tau(\gamma_i)$ . If  $\gamma = \{(\gamma_i, m_i)\}_{i=1}^k$  is an orbit set, then we define the “*symmetric*” *Conley-Zehnder index* (so called because of its motivation from

studying symplectomorphisms of a symmetric product of a surface) as follows:

$$(3.4.1) \quad \tilde{\mu}_\tau(\gamma) = \sum_{i=1}^k \sum_{j=1}^{m_i} \mu_\tau(\gamma_i^j),$$

where  $\gamma_i^j$  is the orbit which multiply covers  $\gamma_i$  with multiplicity  $j$ .

We define the *relative intersection pairing*  $Q_\tau(Z)$  as follows: Using the trivialization  $\tau$ , for each simple orbit  $\gamma_i$  of  $\gamma$  or  $\gamma'$ , fix an identification of each sufficiently small neighborhood  $N(\gamma_i)$  of  $\gamma_i$  with  $\gamma_i \times D^2$ , where  $D^2$  has polar coordinates  $(r, \theta)$ . Let  $\Sigma$  be an oriented embedded surface and  $f : \Sigma \rightarrow [-1, 1] \times M$  be a smooth map which satisfies the following:

- (1)  $f$  maps  $\partial\Sigma$  to  $\{-1, 1\} \times M$ ,  $f|_{\text{int}(\Sigma)}$  is an embedding, and  $f$  is transverse to  $\{-1, 1\} \times M$ .
- (2) For all  $\varepsilon > 0$  sufficiently small,  $f(\Sigma) \cap (\{1 - \varepsilon\} \times M)$  consists of  $m_i$  disjoint circles of type  $\{r = \varepsilon, \theta = \text{const}\}$  in  $N(\gamma_i)$  for all  $i$  (and similarly for  $f(\Sigma) \cap (\{-1 + \varepsilon\} \times M)$ ).
- (3) The composition of  $f$  with the projection  $[-1, 1] \times M \rightarrow M$  is a representative of the class  $Z \in H_2(M, \gamma, \gamma')$ .

We then choose two maps  $f_1, f_2$  satisfying (1)–(3) above, such that they are disjoint on  $\{-1 + \varepsilon, 1 - \varepsilon\} \times M$  and transverse on  $[-1 + \varepsilon, 1 + \varepsilon] \times M$ . Then  $Q_\tau(Z)$  is the signed intersection number of  $f_1$  and  $f_2$  away from  $\{-1, 1\} \times M$ .

Finally, the ECH index  $I_{ECH}(\gamma, \gamma', Z)$  is given by:

$$(3.4.2) \quad I_{ECH}(\gamma, \gamma', Z) = c_\tau(\xi|_Z) + Q_\tau(Z) + \tilde{\mu}_\tau(\gamma) - \tilde{\mu}_\tau(\gamma').$$

See [Hu, Definition 1.5]. We will simply write  $I = I_{ECH}$  for the rest of the paper.

*Remark 3.4.1.* In the sequel [CGH] we introduce an analogous index  $I_{HF}$  in the context of Heegaard Floer homology, and will need to keep track of subscripts.

*Remark 3.4.2.* A finite energy holomorphic map  $u$  with asymptotics  $\gamma$  and  $\gamma'$  defines a relative homology class  $Z \in H_2(M, \gamma, \gamma')$ . Hence we can write  $I(u) = I(\gamma, \gamma', Z)$ .

**3.5. The ECH differential.** We now define the differential  $\partial$  for the ECH chain complex  $ECC(M, \alpha, J)$  with  $J$  regular. First recall that an  $I = 0$  curve  $u$  is called a *connector*; such a curve was shown to be a branched cover (with possibly empty branch locus) of a trivial cylinder  $\mathbb{R} \times \delta$  over a simple orbit  $\delta$ . Writing  $\partial$  as:

$$\partial\gamma = \langle \partial\gamma, \gamma' \rangle \gamma',$$

then  $\langle \partial\gamma, \gamma' \rangle$  is the (mod 2) count of  $u \in \mathcal{M}_J(\gamma, \gamma')$  where  $I(u) = 1$  and every connector component of  $u$  is a trivial cylinder, i.e., is not a branched cover. Here  $I(u) = I(\pi_M \circ u(F))$ , where  $\pi_M : \mathbb{R} \times M \rightarrow M$  is the projection onto the second factor.

We will usually denote a holomorphic curve  $u : (F, j) \rightarrow (\mathbb{R} \times M, J)$  by  $u(F)$ ; this is justified because  $I = 1$  curves are (mostly) embedded.

The map  $\partial$  was shown to satisfy  $\partial^2 = 0$  by Hutchings-Taubes [HT1, HT2]. The homology of the chain complex  $(ECC(M, \alpha, J), \partial)$  is the *embedded contact homology* group  $ECH(M, \alpha, J)$ . It is independent of the choice of contact form  $\alpha$ , the complex structure  $\xi$ , and adapted almost complex structure  $J$ , by the work of Taubes [T2]. Hence we are justified in writing  $ECH(M)$ .

**3.6. Definition of  $\widehat{ECH}(M)$ .** We now define the variant  $\widehat{ECH}(M)$  of  $ECH(M)$ , called the *ECH hat group*. First pick a generic point  $z \in \mathbb{R} \times M$ . Then we define the map:

$$U : ECC(M, \alpha, J) \rightarrow ECC(M, \alpha, J),$$

where  $\langle U\gamma, \gamma' \rangle$  is the (mod 2) count of holomorphic maps  $u : (F, j) \rightarrow (\mathbb{R} \times M, J)$  of ECH index  $I = 2$  from  $\gamma$  to  $\gamma'$  which pass through the point  $z$ . Then  $\widehat{ECH}(M, \alpha, J)$  is defined as the mapping cone of  $U$ . The group  $\widehat{ECH}(M, \alpha, J)$  also has an interpretation as a sutured ECH group, by the work of [CGHH].

#### 4. COBORDISM MAPS AND DIRECT LIMITS

In this section we review the work of Hutchings-Taubes [HT3] on maps on ECH induced by exact symplectic cobordisms, which in turn makes it possible to take direct limits in ECH.

**4.1. Maps induced by cobordisms.** Let  $(M_1, \alpha_1)$  and  $(M_2, \alpha_2)$  be contact 3-manifolds. An *exact symplectic cobordism*  $(X, \omega)$  from  $(M_1, \alpha_1)$  to  $(M_2, \alpha_2)$  is an exact symplectic manifold with boundary  $\partial X = M_1 - M_2$  and symplectic form  $\omega = d\alpha$ , where  $\alpha$  restricts to  $\alpha_1$  on  $M_1$  and  $\alpha_2$  on  $M_2$ .

We write  $ECC^{\leq L}(M, \alpha)$  to denote the subcomplex of  $ECC(M, \alpha)$  generated by orbit sets  $\gamma$  of action  $\mathcal{A}_\alpha(\gamma) \leq L$ , and  $ECH^{\leq L}(M, \alpha)$  to denote the resulting homology group. Given  $L < L'$ , the inclusion of chain complexes  $ECC^{\leq L}(M, \alpha) \subset ECC^{L'}(M, \alpha)$  induces a map

$$i_{L, L'} : ECH^{\leq L}(M, \alpha) \rightarrow ECH^{\leq L'}(M, \alpha)$$

on the level of homology. The following is an immediate consequence of the definition of a direct limit:

$$ECH(M, \alpha) = \lim_{L \rightarrow \infty} ECH^{\leq L}(M, \alpha).$$

The main technical result of [HT3] is the following:

**Theorem 4.1.1** (Hutchings-Taubes). *Let  $(M_1, \alpha_1)$  and  $(M_2, \alpha_2)$  be contact 3-manifolds and  $(X, \omega)$  be an exact symplectic cobordism from  $(M_1, \alpha_1)$  to  $(M_2, \alpha_2)$ . Suppose the contact forms  $\alpha_1, \alpha_2$  are nondegenerate. Then for each positive real number  $L$  there exists a map:*

$$\Phi^L(X, \omega) : ECH^{\leq L}(M_1, \alpha_1) \rightarrow ECH^{\leq L}(M_2, \alpha_2).$$

Moreover, the following are satisfied:

- (i) *Let  $J$  be an almost complex structure on  $X$  which is  $\omega$ -compatible and is adapted to the symplectizations of  $(M_i, \alpha_i)$  at the positive and negative ends. Then  $\Phi^L(X, \omega)$  is induced from a chain map*

$$\Phi^L(X, \omega, J) : ECC^{\leq L}(M_1, \alpha_1) \rightarrow ECC^{\leq L}(M_2, \alpha_2),$$

*which is supported on the  $J$ -holomorphic curves, i.e.,*

$$\langle \Phi^L(X, \omega, J)(\gamma), \gamma' \rangle \neq 0$$

*only if there is a  $J$ -holomorphic curve from  $\gamma$  to  $\gamma'$  in  $X$ .*

- (ii) *The map  $\Phi^L(X, \omega)$  on homology only depends on  $L$  and  $(X, \omega)$ , and not on any auxiliary almost complex structure  $J$  on  $(X, \omega)$ .*
- (iii) *If  $L < L'$ , then the following diagram commutes:*

$$(4.1.1) \quad \begin{array}{ccc} ECH^{\leq L}(M_1, \alpha_1) & \xrightarrow{\Phi^L(X, \omega)} & ECH^{\leq L}(M_2, \alpha_2) \\ \downarrow i_{L, L'} & & \downarrow i_{L, L'} \\ ECH^{\leq L'}(M_1, \alpha_1) & \xrightarrow{\Phi^{L'}(X, \omega)} & ECH^{\leq L'}(M_2, \alpha_2) \end{array}$$

Hence the maps pass to the direct limit:

$$\Phi(X, \omega) : ECH(M_1, \alpha_1) \rightarrow ECH(M_2, \alpha_2).$$

- (iv) *Suppose  $(X, \omega)$  is the composition of exact symplectic cobordisms  $(X_1, \omega_1)$  from  $(M_1, \alpha_1)$  to  $(M', \alpha')$  and  $(X_2, \omega_2)$  from  $(M', \alpha')$  to  $(M_2, \alpha_2)$ , and  $\alpha'$  is nondegenerate. Then*

$$\Phi^L(X, \omega) = \Phi^L(X_2, \omega_2) \circ \Phi^L(X_1, \omega_1).$$

- (v) *If  $c > 0$ , then the following diagram commutes:*

$$(4.1.2) \quad \begin{array}{ccc} ECH^{\leq L}(M_1, \alpha_1) & \xrightarrow{\Phi^L(X, \omega)} & ECH^{\leq L}(M_2, \alpha_2) \\ \downarrow s & & \downarrow s \\ ECH^{\leq cL}(M_1, c\alpha_1) & \xrightarrow{\Phi^{cL}(X, c\omega)} & ECH^{\leq cL}(M_2, c\alpha_2), \end{array}$$

where  $s$  is the canonical rescaling isomorphism.

(vi) If  $X = [0, a] \times M$  and  $\omega = d(e^s \alpha)$ , then

$$\Phi^L(X, \omega) : ECH^{\leq L}(M, e^a \alpha) \rightarrow ECH^{\leq L}(M, \alpha)$$

is equal to the composition

$$ECH^{\leq L}(M, e^a \alpha) \xrightarrow{s} ECH^{\leq e^{-a}L}(M, \alpha) \xrightarrow{i_{e^{-a}L, L}} ECH^{\leq L}(M, \alpha).$$

**4.2. Direct limits.** One consequence of Theorem 4.1.1 is the following theorem:

**Theorem 4.2.1** (Hutchings-Taubes). *Let  $M$  be a closed oriented 3-manifold with a contact form  $\alpha$ , let  $\{f_i\}_{i=1}^\infty$  be a sequence of smooth positive functions such that  $1 \geq f_1 \geq f_2 \dots$  and  $f_i \alpha$  is nondegenerate for each  $i$ , and let  $L_i \rightarrow \infty$  be an increasing sequence of real numbers. Then there is a canonical isomorphism*

$$ECH(M, \alpha) = \lim_{i \rightarrow \infty} ECH^{\leq L_i}(M, f_i \alpha).$$

We can now quantify when it makes sense to take direct limits of a sequence  $\alpha_i$  of isotopic contact forms. Note that, by Gray's theorem, we can write  $\phi_i^*(\alpha_i) = f_i \alpha$  for some positive function  $f_i$  and diffeomorphism  $\phi_i$  isotopic to the identity.

**Definition 4.2.2.** Let  $\alpha$  be a contact form on  $M$ . A sequence  $\{\alpha_i\}_{i=1}^\infty$  of contact forms on  $M$  is *commensurate to  $\alpha$*  if there is a constant  $0 < c < 1$ , diffeomorphisms  $\phi_i$  of  $M$  isotopic to the identity, and functions  $f_i : M \rightarrow \mathbb{R}^{>0}$  such that  $\phi_i^* \alpha_i = f_i \alpha$  and  $c < |f_i|_{C^0} < \frac{1}{c}$ .

A corollary of Theorem 4.2.1 is the following:

**Corollary 4.2.3.** *Let  $\{\alpha_i\}$  be a sequence of contact 1-forms on  $M$  which is commensurate to  $\alpha$  on  $M$  with constant  $0 < c < 1$ . If  $L_i \rightarrow \infty$  is a sequence which satisfies  $L_{i+1} > \frac{1}{c^2} L_i$  for all  $i$ , then we have:*

$$ECH(M) = \lim_{i \rightarrow \infty} ECH^{\leq L_i}(M, \alpha_i).$$

*Proof.* Apply Theorem 4.2.1 to the sequence of functions  $g_n = c^{-2n} f_n$ .  $\square$

## 5. MORSE-BOTT THEORY

In this section we discuss Morse-Bott theory as it applies to our context. In particular, we explain how to use Theorem 4.2.1 to justify the Morse-Bott arguments which populate this paper. For a detailed discussion of Morse-Bott theory in contact homology, the reader is referred to Bourgeois [Bo1, Bo2].

**5.1. Morse-Bott contact forms.** Let  $\alpha$  be a *Morse-Bott contact form* on  $M$ . For the purposes of this paper, this means that all the orbits either are isolated and nondegenerate, or come in  $S^1$ -families and are nondegenerate in the Morse-Bott sense. Denote the isolated simple orbits by  $\gamma_j$  and the Morse-Bott families of simple orbits by  $\mathcal{N}_i$ . The Morse-Bott torus corresponding to  $\mathcal{N}_i$  will be written as  $T_{\mathcal{N}_i} = \cup_{x \in \mathcal{N}_i} x$ . Let  $\mathcal{P}_{MB} = (\cup_j \{\gamma_j\}) \cup (\cup_i \mathcal{N}_i)$  be the set of simple orbits of  $R_\alpha$ . On each  $\mathcal{N}_i \simeq S^1$ , we pick a Morse function  $g_i : \mathcal{N}_i \rightarrow \mathbb{R}$ , which we assume to have two critical points  $e_i$  and  $h_i$ . An *orbit set*  $\gamma$  for the Morse-Bott contact form  $\alpha$  is an orbit set constructed from  $\mathcal{P} = (\cup_j \{\gamma_j\}) \cup (\cup_i \{h_i, e_i\})$ , where  $h_i$  is treated as a hyperbolic orbit and  $e_i$  is treated as an elliptic orbit.

The Morse index of  $e_i$  with respect to the function  $g_i$  can be higher or lower than that of  $h_i$ , depending on the type of Morse-Bott family. Let  $\{v_1, v_2\}$  be an oriented basis for  $\xi$  at some point  $p$  of  $T_{\mathcal{N}_i}$  so that  $v_1$  is transverse to  $T_{\mathcal{N}_i}$  and  $v_2$  is tangent to  $T_{\mathcal{N}_i}$ . Suppose the derivative  $\xi_p \rightarrow \xi_p$  of the first return map of the Reeb flow is given by  $\begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix}$  with respect to  $\{v_1, v_2\}$ . (Here a vector  $v = a_1 v_1 + a_2 v_2$  is written as a column vector.) If  $a > 0$ , then, after perturbing the Morse-Bott family so that  $e_i$  becomes elliptic and  $h_i$  becomes hyperbolic (see Section 5.5 for more details), their Conley-Zehnder indices, measured with respect to the framing induced by  $T_{\mathcal{N}_i}$ , become  $\mu(e_i) = 1$  and  $\mu(h_i) = 0$ . Such a Morse-Bott family  $\mathcal{N}_i$  is said to be *positive*. On the other hand, if  $a < 0$ , then  $\mu(e_i) = -1$ ,  $\mu(h_i) = 0$ , and  $\mathcal{N}_i$  is *negative*.

**5.2. Morse-Bott regularity.** Let  $J$  be an almost complex structure on  $\mathbb{R} \times M$  which is adapted to the Morse-Bott contact form  $\alpha$ . Although the notation is a bit cumbersome, consider the moduli space

$$\mathcal{M}_J(\gamma_1^+, \dots, \gamma_{i_1}^+; \mathcal{N}_1^+, \dots, \mathcal{N}_{i_2}^+; \gamma_1^-, \dots, \gamma_{i_3}^-; \mathcal{N}_1^-, \dots, \mathcal{N}_{i_4}^-),$$

abbreviated  $\mathcal{M}_J(\gamma^+, \mathcal{N}^+, \gamma^-, \mathcal{N}^-)$ , of  $J$ -holomorphic maps  $u$  in  $\mathbb{R} \times M$  which have positive ends at  $\gamma_1^+, \dots, \gamma_{i_1}^+, \tilde{\gamma}_1^+, \dots, \tilde{\gamma}_{i_2}^+$  and negative ends at  $\gamma_1^-, \dots, \gamma_{i_3}^-, \tilde{\gamma}_1^-, \dots, \tilde{\gamma}_{i_4}^-$ , where  $\tilde{\gamma}_i^\pm$  is in the Morse-Bott family  $\mathcal{N}_i^\pm$ . For the moment we do not assume that  $\mathcal{N}_i^\pm$  consist of simple orbits.

We say that  $J$  is *Morse-Bott regular* if, for all data  $\gamma^+, \mathcal{N}^+, \gamma^-, \mathcal{N}^-$  and  $u \in \mathcal{M}_J(\gamma^+, \mathcal{N}^+, \gamma^-, \mathcal{N}^-)$  which have no multiply-covered components,  $\mathcal{M}_J(\gamma^+, \mathcal{N}^+, \gamma^-, \mathcal{N}^-)$  is transversely cut out (and hence is a manifold) near  $u$ . Such  $J$  are dense by a result of Bourgeois [Bo2].

**5.3. Morse-Bott buildings.** In this subsection we take  $\mathcal{N}_i$  to be Morse-Bott families of simple orbits.



We make the following (somewhat nonstandard) definition. Due to the simple nature of the Morse-Bott families  $\mathcal{N}_i$ , we do not need to consider Morse-Bott buildings with more stories.

**Definition 5.3.1.** Given two orbit sets  $\gamma$  and  $\gamma'$  constructed from  $\mathcal{P}$ , a *Morse-Bott building  $\tilde{u}$  from  $\gamma$  to  $\gamma'$  without connector components* is an object of one of two types:

- (A) If  $\gamma \neq \gamma'$  are critical points of  $\mathcal{N}_i$ , then a gradient trajectory of  $g_i$  from  $\gamma$  to  $\gamma'$ ;
- (B) A  $J$ -holomorphic curve  $u$  in  $\mathbb{R} \times M$  without connector components from  $\gamma_0$  to  $\gamma'_0$ , where the orbit sets  $\gamma_0$  and  $\gamma'_0$  are constructed from  $\mathcal{P}_{MB}$ . If an end  $x^k$ ,  $x \in \mathcal{N}_i$ , is not in  $\mathcal{P}$ , then  $x^k$  is augmented by a gradient trajectory  $\delta_x$  in  $\mathcal{N}_i$ . The gradient trajectory  $\delta_x$  flows from  $y \in \mathcal{P}$  to  $x$  if  $x$  is a positive end and from  $x$  to  $y \in \mathcal{P}$  if  $x$  is a negative end. The orbit sets  $\gamma$  and  $\gamma'$  are obtained from  $\gamma_0$  and  $\gamma'_0$  by replacing  $x^k$  by the corresponding  $y^k$ . The curve  $u$  will be called the *principal part* of  $\tilde{u}$ .

A Morse-Bott building  $\tilde{u}$  admits a decomposition  $\tilde{u}_0 \cup \tilde{u}_1$ , where  $\tilde{u}_0$  is a connector over orbits in  $\mathcal{P}$  and  $\tilde{u}_1$  is a Morse-Bott building without trivial components.

A Morse-Bott building  $\tilde{u} = \tilde{u}_0 \cup \tilde{u}_1$  such that  $\tilde{u}_1$  satisfies (A) (resp. (B)) of Definition 5.3.1 will be called a *Morse-Bott building of type A* (resp. *type B*). The set of Morse-Bott buildings  $\tilde{u}$  from  $\gamma$  to  $\gamma'$  will be denoted by  $\mathcal{M}_J^{MB}(\gamma, \gamma')$ .

**Definition 5.3.2.** A *simply-covered* Morse-Bott building is a Morse-Bott building which is either (i) of type A or (ii) of type B and whose principal part is simply-covered. A *multiply-covered* Morse-Bott building is a Morse-Bott building which is not simply-covered.

**Definition 5.3.3.** A Morse-Bott building  $\tilde{u} \in \mathcal{M}_J^{MB}(\gamma, \gamma')$  is *nice* if it is simply-covered and has no end which multiply covers a simple orbit of  $\mathcal{N}_i$ , except for the following:

- (+)  $\mathcal{N}_i$  is positive and the end is a negative end which multiply covers  $e_i$ ;
- (−)  $\mathcal{N}_i$  is negative and the end is a positive end which multiply covers  $e_i$ .

*Remark 5.3.4.* Niceness allows for the possibility of several ends, each of which is asymptotic to a cylinder over a simple  $x \in \mathcal{N}_i$ .

**5.4. Morse-Bott chain complex.** In this subsection we define the ECH index of a Morse-Bott building and describe the ECH Morse-Bott chain complex.

**Definition 5.4.1.** The *ECH index*  $I(\tilde{u})$  of a Morse-Bott building  $\tilde{u}(F)$  is  $I(\gamma, \gamma', A)$ , where:

- If  $\tilde{u}$  is of type A, then  $A$  is the union of an annulus corresponding to the gradient trajectory and the projection of the connectors  $\tilde{u}_0$  to  $M$ .
- If  $\tilde{u}$  is of type B, then  $A$  is obtained from projecting the principal part  $u$  and the connectors  $\tilde{u}_0$  to  $M$  and gluing the annuli (or  $k$ -fold covers of them) corresponding to the covers of the augmenting gradient trajectories.
- The Conley-Zehnder indices of  $\gamma$  and  $\gamma'$  are computed with the convention that  $\mu_\tau(e_i^j) = 1$  and  $\mu_\tau(h_i) = 0$  if  $\mathcal{N}_i$  is a positive Morse-Bott torus, and  $\mu_\tau(h_i) = 0$  and  $\mu_\tau(e_i^j) = -1$  if  $\mathcal{N}_i$  is a negative Morse-Bott torus. Here  $\tau$  is the trivialization defined by  $\mathcal{N}_i$ .

The ECH index computed with this definition coincides with the limit of ECH indices computed with respect to nondegenerate perturbations  $f_\alpha$  of the form  $\alpha$ , as the perturbing function  $f$  approaches 1.

The Fredholm index  $\text{ind}(\tilde{u})$  of a Morse-Bott building  $\tilde{u}$  can be defined by modifying the definition of the usual Fredholm index in a manner similar to that of  $I(\tilde{u})$ . Here our convention is that the Fredholm index takes into account the dimensions of the moduli space and the automorphism group of the domain of the map.

Let  $J$  be a Morse-Bott regular almost complex structure. Then the Morse-Bott chain complex  $(ECC_{MB}^{\leq L}(M, \alpha, J), \partial_{MB})$  is generated by orbit sets constructed from  $\mathcal{P}$  and the differential counts Morse-Bott buildings with  $I(\tilde{u}) = 1$ . By following proposition,  $\partial_{MB}$  only counts nice Morse-Bott buildings.

**Proposition 5.4.2.** *Let  $J$  be a Morse-Bott regular almost complex structure and let  $\gamma, \gamma'$  be orbit sets for  $ECC_{MB}^{\leq L}(M, \alpha, J)$ . Then an  $I = 1$  Morse-Bott building  $\tilde{u}$  is a nice Morse-Bott building.*

*Proof.* A Morse-Bott building  $\tilde{u}$  is not nice if it is either (a) simply-covered, but has an end which multiply covers an orbit in some Morse-Bott torus (with the exception of  $(+)$  and  $(-)$  of Definition 5.3.3), or (b) multiply-covered. We will show that either case implies  $I > 1$ .

(a) We first eliminate Morse-Bott buildings  $\tilde{u}$  that are simply-covered but have an end which multiply covers an orbit in some Morse-Bott torus, with the exception of  $(+)$  and  $(-)$ . The principal part  $u$  of  $\tilde{u}$  satisfies  $\text{ind}(u) \geq 1$  by the Morse-Bott regularity of  $J$ . To the curve  $u$  we would like to “pre-glue” covers of cylinders corresponding to gradient trajectories, possibly with branch points. The resulting pre-glued curve will be called  $u^\#$ .

Suppose  $u$  has exactly one positive end on a cover of a positive  $\mathcal{N}_i$  and it multiply covers  $x \in \mathcal{N}_i$  with multiplicity  $k > 1$ . Here  $x \neq h_i$  since hyperbolic orbits have multiplicity 1. By the “incoming/outgoing partition” considerations of [Hu], the partition for  $e_i^k$  at the positive end must be  $(1, \dots, 1)$ . This is due to the fact that the linearized return map along  $e_i$  is an arbitrarily small positive rotation if we take  $f_j$  to be arbitrarily close to 1. In other words, we need to take a branched cover of a cylinder from  $e_i$  to  $x$  so that there are  $k$  positive ends and one negative end which covers  $x^k$ .

By the usual Fredholm index formula, the branched cover  $C$  contributes

$$(5.4.1) \quad \text{ind}(C) = -\chi(C) + k \cdot \mu(e_i) - \mu(e_i^k)$$

$$(5.4.2) \quad = -\chi(C) + k \cdot 1 - 1 = -\chi(C) + (k - 1),$$

and  $\text{ind}(C) \geq 2$  if  $k > 1$ . By the additivity of the Fredholm index, the pre-glued curve  $u^\#$  satisfies  $\text{ind}(u^\#) \geq 3$ . Moreover, by the ECH index inequality [Hu, Theorem 1.7], we have  $I(u^\#) \geq \text{ind}(u^\#) \geq 3$ .

Note that  $u^\#$  is not quite holomorphic so the ECH index inequality does not apply verbatim; however, the proof carries over with very little modification. The key point is that our pre-glued curve  $u^\#$  satisfies the following adjunction inequality with singularities [Hu, Remark 3.2]:

$$(5.4.3) \quad c_\tau(\xi|_{u^\#}) = \chi(u^\#) + w_\tau(u^\#) + Q_\tau(u^\#) - 2\delta(u^\#),$$

where  $w_\tau(u^\#)$  is the *writhe* as defined in [Hu, Section 3.1] and the count  $\delta(u^\#)$  of singularities is *nonnegative*. The nonnegativity of  $\delta(u^\#)$  can be seen by intersecting  $u^\#$  and a translate  $u_{s_0}^\#$  of  $u^\#$  in the  $s$ -direction by  $s_0 \gg 0$ . Since the intersections can be made to take place only on holomorphic part (i.e., between the principal part  $u$  and its translate  $u_{s_0}$ ), it follows that  $\delta(u^\#) \geq 0$ .

The above calculation can be easily generalized to the case where there are more ends of  $u$  on a cover of  $\mathcal{N}_i$  and when  $u$  has a negative end on a cover of a negative  $\mathcal{N}_i$ .

(b) Now we eliminate multiply-covered Morse-Bott buildings. The ECH differential in the non-Morse-Bott (i.e., usual) case does not count multiply-covered curves besides connectors, since multiply-covered curves which are not connectors have ECH index  $I > 1$  by [Hu, Theorem 8.1]. We make a similar argument for Morse-Bott buildings. If  $\tilde{u}$  is a multiply-covered Morse-Bott building, then the principal part  $u$  multiply covers some simple  $u'$  with multiplicity  $m$ . If  $u'$  is nice, then we can pre-glue trivial cylinders corresponding to gradient trajectories onto  $u'$  to obtain  $(u')^\#$ . By the regularity of  $J$ ,  $\text{ind}((u')^\#) \geq 1$ . By the ECH index inequality we have:

$$I((u')^\#) \geq \text{ind}((u')^\#) \geq 1.$$

Now [HT1, Prop. 7.15] states that if  $v$  is an  $m$ -fold cover of a simple holomorphic curve  $v'$ , then  $I(v) \geq m \cdot I(v')$ . Hence,

$$I(\tilde{u}) \geq m \cdot I((u')^\#) \geq m > 1.$$

When  $u'$  is simple but not nice, pre-gluing branched covers of cylinders corresponding to gradient trajectories further increases the Fredholm index and hence the ECH index.  $\square$

**5.5. Direct limit arguments.** Let  $L_i \rightarrow \infty$  be an increasing sequence. Let  $\mathcal{N}_1, \dots, \mathcal{N}_{n(i)}$  be the Morse-Bott tori consisting of simple orbits with  $\alpha$ -action  $< L_i$ . Then there exists a function  $f_i$  which satisfies the following:

- (1)  $f_i = 1 + \varepsilon_i g_i$ , where  $\varepsilon_i > 0$  is arbitrarily small and  $g_i$  is supported on arbitrarily small neighborhoods of  $T_{\mathcal{N}_1} \cup \dots \cup T_{\mathcal{N}_{n(i)}}$  and extends the Morse function  $g_i : \mathcal{N}_i \rightarrow \mathbb{R}$ ; in particular,  $f_i$  can be assumed to be  $C^k$ -close to 1 for  $k \gg 0$ .
- (2) The Reeb orbits of  $f_i \alpha$  which have  $f_i \alpha$ -action  $\leq L_i$  are nondegenerate.
- (3) Each  $\mathcal{N}_i$  is perturbed into the pair  $e_i$  and  $h_i$ ; all multiples  $e_i^k$  and  $h_i^k$  of action  $\mathcal{A}_{f_i \alpha}(\gamma) < L_i$  have Conley-Zehnder indices 1 and 0 if  $\mathcal{N}_i$  is positive and  $-1$  and 0 if  $\mathcal{N}_i$  is negative.
- (4) All other orbits  $\gamma$  which are created have action  $\mathcal{A}_{f_i \alpha}(\gamma) > L_i$ .

According to Corollary 4.2.3,

$$(5.5.1) \quad ECH(M) = \lim_{i \rightarrow \infty} ECH^{\leq L_i}(M, f_i \alpha).$$

We will now work on  $ECC^{\leq L_i}(M, f_{ij} \alpha, J_{ij})$ , where:

- (A1)  $L_i$  is fixed;
- (A2)  $f_{ij}$  satisfies (1)–(4) above and approaches 1 as  $j \rightarrow \infty$ ;
- (A3)  $g_{ij}$  has support which approaches  $T_{\mathcal{N}_1} \cup \dots \cup T_{\mathcal{N}_{n(i)}}$  as  $j \rightarrow \infty$ ; and
- (A4)  $J_{ij}$  is a regular almost complex structure on  $\mathbb{R} \times M$  adapted to  $f_{ij} \alpha$ .

For convenience we suppress the subscript  $i$ . Also let  $J$  be a Morse-Bott regular almost complex structure on  $\mathbb{R} \times M$  adapted to  $\alpha$ .

**Definition 5.5.1.** A  $J_j$ -holomorphic curve  $u_j \in \mathcal{M}_{J_j}(\gamma, \gamma')$  is *nice* if it is simply-covered and has no end which multiply covers  $e_i$  or  $h_i$ , except for the following:

- (+)  $e_i$  belongs to a positive  $\mathcal{N}_i$  and the end is a negative end which multiply covers  $e_i$ ;
- (−)  $e_i$  belongs to a negative  $\mathcal{N}_i$  and the end is a positive end which multiply covers  $e_i$ .

The following result was proven in Bourgeois [Bo2]; we have slightly rephrased it for our purposes:

**Theorem 5.5.2** (Bourgeois). *If  $\gamma, \gamma'$  are orbit sets for  $ECC^{\leq L}(M, f_j\alpha, J_j)$ , then the following hold:*

- (1) *Any sequence  $u_j \in \mathcal{M}_{J_j}(\gamma, \gamma')$  has a subsequence which converges to a Morse-Bott building  $\tilde{u} \in \mathcal{M}_J^{MB}(\gamma, \gamma')$ .*
- (2) *For  $f_j$  sufficiently close to 1, the algebraic count of  $I(u_j) = 1$  curves in  $\mathcal{M}_{J_j}(\gamma, \gamma')$  is the same as the algebraic count of  $I(\tilde{u}) = 1$  Morse-Bott buildings in  $\mathcal{M}_J^{MB}(\gamma, \gamma')$ .*

Here, an “algebraic count” is the mod 2 count of holomorphic curves of Fredholm index 1 after quotienting by  $\mathbb{R}$ -translations.

*Proof.* We only need to explain the role of  $I = 1$  curves in (2). An  $I = 1$  curve  $u_j \in \mathcal{M}_{J_j}(\gamma, \gamma')$  can be shown to be nice by incoming/outgoing partition considerations, and an  $I = 1$  curve  $\tilde{u} \in \mathcal{M}_J^{MB}(\gamma, \gamma')$  is nice by Proposition 5.4.2. (Moreover,  $I = 1$  implies  $\text{ind} = 1$ .) Hence the principal part  $u$  of  $\tilde{u}$  is glued to non-multiply-covered cylinders corresponding to the augmenting gradient trajectories. This is explained in [Bo2] and does not require any abstract perturbations.  $\square$

**Proposition 5.5.3.**

$$ECH^{\leq L}(M, f_j\alpha, J_j) \simeq ECH_{MB}^{\leq L}(M, \alpha, J).$$

*Proof.* The isomorphism follows from the description of the holomorphic curves counted in  $\partial_{MB}$  given in Proposition 5.4.2, together with Theorem 5.5.2.  $\square$

By passing to direct limits, we obtain:

**Corollary 5.5.4.**

$$ECH(M, f_j\alpha, J_j) \simeq ECH_{MB}(M, \alpha, J).$$

**5.6. The winding number.** Let us recall the *winding number* from [HWZ]: Given a contact structure  $(M, \xi = \ker \alpha)$ , an  $\alpha$ -adapted almost complex structure  $J$  on  $\mathbb{R} \times M$ , and a  $J$ -holomorphic curve  $u(F)$  between orbits sets, the *winding number*  $\text{wind}_\pi(u(F))$  is an algebraic count of the zeros of the section:

$$s : F \rightarrow \text{Hom}_{\mathbb{C}}(TF, u^*\xi).$$

Here  $s$  is obtained by composing

$$TF \xrightarrow{u_*} T(\mathbb{R} \times M) \xrightarrow{(\pi_M)^*} TM \xrightarrow{\pi} \xi,$$

where  $\pi_M : \mathbb{R} \times M \rightarrow M$  is the projection onto the second factor and  $\pi$  is the projection along the Reeb vector field  $R_\alpha$ .

In [HWZ], Hofer-Wysocki-Zehnder prove that  $\text{wind}_\pi(u(F))$  is finite. An immediate corollary is the following lemma:

**Lemma 5.6.1.** *The projection  $\pi_M(u(F))$  is transverse to  $R_\alpha$  away from a finite number of points on  $F$ .*

**5.7. Blocking lemma.** Fix  $L > 0$ . Let  $J_j$  be a regular almost complex structure on  $\mathbb{R} \times M$  adapted to the contact form  $f_j\alpha$  and let  $u_j$  be a holomorphic curve in  $(\mathbb{R} \times M, J_j)$  whose ends have  $f_j\alpha$ -action  $\leq L$ .

Consider a Morse-Bott torus  $T \simeq \mathbb{R}^2/\mathbb{Z}^2$  for  $(M, \alpha)$ , where  $T$  is foliated by Reeb orbits of  $R_\alpha$  directed by  $(0, 1)$ . We assume that  $T$  is a negative Morse-Bott torus and there is a small neighborhood  $T^2 \times [-\varepsilon, \varepsilon] \subset M$  of  $T = T^2 \times \{0\}$  such that the following hold:

- The tori  $T_s = T^2 \times \{s\}$ ,  $s \in [-\varepsilon, \varepsilon]$ , are foliated by Reeb orbits of  $R_\alpha$ .
- $R_\alpha$  on  $T_\varepsilon$  is directed by  $(-1, N)$  and  $R_\alpha$  on  $T_{-\varepsilon}$  is directed by  $(1, N)$ , where  $N \gg 0$ .
- There are no Reeb orbits of  $R_\alpha$  of action  $\leq L$  on  $T_s$  for  $s \in [-\varepsilon, \varepsilon] - \{0\}$ .

We say that  $u_j(F_j)$  *crosses*  $T^2 \times [-\varepsilon, \varepsilon]$  if there is a connected component of

$$u_j(F_j) \cap (\mathbb{R} \times T^2 \times [-\varepsilon, \varepsilon]),$$

which intersects both  $\mathbb{R} \times T_{-\varepsilon}$  and  $\mathbb{R} \times T_\varepsilon$ .

We now consider

$$\delta_{\pm\varepsilon}^j = \pi_M(u_j(F_j)) \cap T_{\pm\varepsilon},$$

where  $\delta_{\pm\varepsilon}^j$  is given the boundary orientation of  $\pi_M(u_j(F_j)) \cap (T^2 \times [-\varepsilon, \varepsilon])$ . (Here  $\pi_M : \mathbb{R} \times M \rightarrow M$ , as before.) Since  $\pi_M(u_j(F_j))$  is transverse to  $R_{f_j\alpha}$  away from a finite number of points on  $F_j$  by Lemma 5.6.1, it follows that  $\delta_{\pm\varepsilon}^j$  is immersed away from a finite number of points, and hence can be assigned a slope in  $T_{\pm\varepsilon}$ .

The following is a “blocking lemma” which prevents certain holomorphic curves from crossing neighborhoods of Morse-Bott tori, and is used extensively throughout the paper. We state it for negative Morse-Bott tori; the version for positive Morse-Bott tori is analogous, after replacing negative ends by positive ends. If we want to specify the contact form  $\alpha$ , we say “ $\alpha$ -Morse-Bott torus”. Also, for simplicity we assume there is only one Morse-Bott torus.

**Lemma 5.7.1** (Blocking lemma). *Let  $T$  be a negative  $\alpha$ -Morse-Bott torus whose orbits are directed by  $(0, 1)$  and let  $T^2 \times [-\varepsilon, \varepsilon] \subset M$  be a neighborhood of  $T = T_0$  as described above. If (A1)–(A4) of Section 5.5 are satisfied (and the index  $i$  suppressed), then for sufficiently large  $j$ :*

- (1) *No holomorphic curve  $u_j(F_j)$  crosses  $T^2 \times [-\varepsilon, \varepsilon]$  if  $\delta_\varepsilon^j$  is a union of curves of slope  $\infty$ .*



- (2) If  $u_j(F_j)$  nontrivially intersects  $\mathbb{R} \times T_\varepsilon$  so that  $\delta_\varepsilon^j$  is a union of curves of slope  $\infty$ , then  $u_j(F_j)$  must be negatively asymptotic to orbits  $h$  or  $e$ , obtained by perturbing  $T$ .

*Proof.* Assume that  $j$  is sufficiently large so that  $g_j$  has support in  $\text{int}(T^2 \times [-\varepsilon/2, \varepsilon/2])$ . In particular,  $R_\alpha = R_{f_j\alpha}$  on  $T_{\pm\varepsilon}$ .

(1) Assume on the contrary that there exists  $u_j(F_j)$  which crosses  $T^2 \times [-\varepsilon, \varepsilon]$ . Then there is a component  $K$  of  $u_j(F_j) \cap (\mathbb{R} \times T^2 \times [-\varepsilon, \varepsilon])$  which has ends on  $\mathbb{R} \times T_{-\varepsilon}$  and  $\mathbb{R} \times T_\varepsilon$ . Without loss of generality, assume that  $\partial(\pi_M(K)) = \delta_\varepsilon^j \cup \delta_{-\varepsilon}^j$  and each boundary component has slope  $\infty$ . By the positivity of intersections in dimension four and the fact that  $T_\varepsilon$  is foliated by Reeb orbits of  $R_{f_j\alpha}$  that are directed by  $(-1, N)$ ,  $\delta_\varepsilon^j$  must be directed by  $(0, 1)$ . Similarly,  $\delta_{-\varepsilon}^j$  must be directed by  $(0, 1)$ . This contradicts the fact that  $[\delta_\varepsilon^j] + [\delta_{-\varepsilon}^j] = 0 \in H_1(T)$  since it is a boundary.

(2) is an immediate corollary of (1).  $\square$

## 6. ECH FOR MANIFOLDS WITH TORUS BOUNDARY

In this section we define several ECH groups on a compact manifold  $M$  with torus boundary  $T = \partial M$ . Fix an oriented identification  $T \simeq \mathbb{R}^2/\mathbb{Z}^2$  so that we can refer to slopes of essential curves on  $T$ . Let  $\alpha$  be a contact form on  $M$  such that  $T$  is foliated by Reeb orbits of slope  $r$ . If  $r$  is rational, we assume that  $T$  is Morse-Bott. All ECH groups on  $M$  and  $\text{int}(M)$  are computed using a  $C^k$ -small perturbation of  $\alpha$  (for an arbitrary large  $k$ ) so that all Reeb orbits in  $\text{int}(M)$  are nondegenerate. Let  $J$  be a Morse-Bott regular almost complex structure on  $\mathbb{R} \times M$  adapted to  $\alpha$ .

**6.1. Definitions.** We introduce several ECH groups:

1.  $ECH(\text{int}(M), \alpha)$ . The ECH chain group  $ECC(\text{int}(M), \alpha)$  is generated by orbit sets whose orbits lie in the interior of  $M$ . In particular, we are discarding the Morse-Bott family of orbits on  $T$  if  $r$  is rational. The differential  $\partial$  is the usual one, i.e., counts holomorphic curves of ECH index  $I(\gamma, \gamma') = I(\gamma, \gamma', Z) = 1$  in  $\mathbb{R} \times \text{int}(M)$  whose connector components are trivial cylinders.

Since  $\text{int}(M)$  is not closed, we need to verify the following:

**Lemma 6.1.1.**  $\partial^2 = 0$ .

*Proof.* Suppose first that  $r$  is rational. Let  $u_i, i = 1, 2, \dots$ , be a sequence of holomorphic curves in  $\mathbb{R} \times \text{int}(M)$  from  $\gamma$  to  $\gamma'$ . Since  $u_i$  does not intersect  $\mathbb{R} \times \delta$ , where  $\delta$  is any orbit of the Morse-Bott family on  $M$ , a limiting curve of the sequence  $u_i$  also cannot intersect  $\mathbb{R} \times \delta$ . This is due to the positivity of intersections of  $J$ -holomorphic curves in dimension four. Hence a 1-parameter family of holomorphic curves from  $\gamma$  to  $\gamma'$  in  $\mathbb{R} \times \text{int}(M)$  cannot



exit from  $\mathbb{R} \times \partial M$ , and the arguments used in [HT1, HT2] to prove  $\partial^2 = 0$  carry over to our setting.

Now suppose  $r$  is irrational. Suppose a sequence  $u_i$  of holomorphic curves from  $\gamma$  to  $\gamma'$  in  $\mathbb{R} \times \text{int}(M)$  converges to a curve  $u$  that intersects  $\mathbb{R} \times T$ . By Lemma 5.6.1,  $u$  intersects  $\mathbb{R} \times T$  only at a finite number of points. It is easy to find a closed curve  $\gamma \subset T$  whose slope is close to  $r$ , which closely approximates a Reeb trajectory of slope  $r$ , and which is tangent to the Reeb vector field in the neighborhood of points where  $\pi_M \circ u$  intersects  $T$ . (Here  $\pi_M$  is the projection  $\mathbb{R} \times M \rightarrow M$ , as usual.) Then  $\mathbb{R} \times \gamma$  intersects  $u$  positively, and the above argument for  $r$  rational carries over.  $\square$

2a.  $ECH(M, \alpha)$  for  $r$  irrational. This is defined to be  $ECH(\text{int}(M), \alpha)$ .

2b.  $ECH(M, \alpha)$  for  $r$  rational. Let  $\mathcal{N}$  be the set of Reeb orbits on  $T$ . The set  $\mathcal{N}$  comes with distinguished orbits  $e, h$  which become elliptic and hyperbolic after a suitable perturbation. Writing  $\mathcal{P}$  for the set of simple orbits in  $\text{int}(M)$ ,  $ECC(M, \alpha)$  is the chain complex which is generated by orbit sets constructed from  $\mathcal{P} \cup \{h, e\}$  and whose differential counts Morse-Bott buildings of ECH index 1 in  $\mathbb{R} \times M$  (which are nice by Proposition 5.4.2).

3.  $ECH^\flat(M, \alpha)$ . The chain complex  $ECC^\flat(M, \alpha)$  is generated by orbit sets which are constructed from  $\mathcal{P} \cup \{e\}$ . As in the case of  $ECC(M, \alpha)$ , if  $\mathcal{N}$  is a negative Morse-Bott family, no Morse-Bott building  $\tilde{u}$  in  $\mathbb{R} \times M$  besides trivial cylinders can have  $e$  at the positive end. Hence the differential can be defined by counting Morse-Bott buildings of ECH index 1 in  $\mathbb{R} \times M$ , whose orbit sets are constructed from  $\mathcal{P} \cup \{e\}$ .

The verification of  $\partial^2 = 0$  needs one extra consideration: An index 2 family of  $J$ -holomorphic curves in  $\mathbb{R} \times M$  which involves  $e$  at the negative end can break into a nice Morse-Bott building  $\tilde{u}$  which involves  $h$  at the negative end. This would be a problem since orbit sets containing  $h$  are not in the chain complex  $ECC^\flat(M, \alpha)$ . However, since there are *two* gradient trajectories from  $h$  to  $e$ , the Morse-Bott building  $\tilde{u}$  can be glued onto each of the two gradient trajectories and  $\partial^2 = 0$  holds even when we discard orbit sets which contain  $h$ .

If  $\mathcal{N}$  is a positive Morse-Bott family, then  $e$  can only be at the positive end of a  $J$ -holomorphic curve in  $\mathbb{R} \times M$ , and the proof of  $\partial^2 = 0$  remains the same with the obvious modifications.

4.  $ECH^\sharp(M, \alpha)$ . The chain complex  $ECC^\sharp(M, \alpha)$  is generated by orbit sets which are constructed from  $\mathcal{P} \cup \{h\}$ , and its differential counts ECH index 1 Morse-Bott buildings which are asymptotic to orbit sets constructed from  $\mathcal{P} \cup \{h\}$ .

**6.2. Well-definition.** In this subsection we prove that  $ECH(M, \alpha)$  is independent of the choice of  $\alpha$ , provided the slope  $r$  is irrational. (The verification in the other cases will be omitted.)

Let  $\alpha_1$  and  $\alpha_2$  be contact forms on  $M$  which agree on a small neighborhood  $T^2 \times [-\varepsilon, 0] \simeq (\mathbb{R}^2/\mathbb{Z}^2) \times [-\varepsilon, 0]$  of  $T = T_0$  with coordinates  $(\theta, t, y)$ , where  $\varepsilon$  is a small positive number. Let us write  $\alpha'$  for  $\alpha_1 = \alpha_2$  on  $T^2 \times [-\varepsilon, 0]$ . Assume that  $T_s$ ,  $s \in [-\varepsilon, 0]$ , is tangent to the Reeb vector field  $R_{\alpha'}$  and the slope of  $R_{\alpha'}$  along  $T$  is a *negative irrational number*  $r$ . (The sign of the slope can be arranged by choosing a suitable oriented identification of  $T$  with  $\mathbb{R}^2/\mathbb{Z}^2$ .) For simplicity also assume that the slope of the characteristic foliation of  $\xi_i = \ker \alpha_i$  is zero along  $T$ .

**Claim 6.2.1.** *Given  $L > 0$ , there is a contact form  $\alpha'(L) = g(y)d\theta + f(y)dt$  on  $T^2 \times [0, \frac{1}{2}]$  with coordinates  $(\theta, t, y)$  which extends  $\alpha'$  on  $T^2 \times [-\varepsilon, 0]$  such that the following hold:*

- *all the closed orbits of  $R = R_{\alpha'(L)}$  have  $\alpha'(L)$ -action  $\geq L$ ;*
- *$(f(0), g(0)) = (c, 0)$  for some  $c > 0$  and  $(f'(0), g'(0))$  has positive irrational slope  $-r$ ;*
- *$(f(\frac{1}{2}), g(\frac{1}{2})) = (1, N)$  for some  $N \gg 0$ .*

*Proof.* Let  $J$  be the standard complex structure,  $\cdot$  be the standard inner product, and  $|\cdot|$  be the standard Euclidean norm on  $\mathbb{R}^2$ . By Lemma 2.3.1 the Reeb vector field  $R$  can be written as:

$$R = \frac{(-f', g')}{(-f', g') \cdot (g, f)}.$$

If we write  $v = (f, g)$ , then  $Jv' = (-g', f')$  and

$$|R| = \left| \frac{(-g', f')}{(-g', f') \cdot (f, g)} \right| = \left| \frac{Jv'}{Jv' \cdot v} \right| = \frac{1}{|v| |\sin \vartheta|},$$

where  $\vartheta$  is the angle between  $v$  and  $v'$ , in view of the fact that  $Jv'$  and  $v'$  are orthogonal.

We then define  $v(y)$  to be an arbitrarily long line segment of irrational slope  $-r$  for  $y \in [0, y_0]$ , where  $y_0 > 0$  is small. In particular, there is no closed Reeb orbit in  $T^2 \times [0, y_0]$ .

On  $[0, y_0]$ , the norm  $|R|$  is a constant, say  $\frac{1}{K}$ . Pick  $-r' > -r$  so that all the shortest integer vectors with slope between  $-r$  and  $-r'$  have norm  $> \frac{L}{K}$ . We then extend  $v(y)$  to  $[y_0, 2y_0]$  so that (i)  $v'(y)$  increases monotonically from  $-r$  to  $-r'$  and (ii)  $v([y_0, 2y_0])$  is short. Then  $|v(2y_0)| \approx |v(y_0)|$  but  $|\sin \vartheta(2y_0)| \gg |\sin \vartheta(y_0)|$ . Hence  $|R(2y_0)| \ll \frac{1}{K}$ . Moreover, on the interval  $[y_0, 2y_0]$ , the actions of the closed orbits are bounded below by  $K \cdot \frac{L}{K} = L$ . We can now extend  $v(y)$  on  $[2y_0, \frac{1}{2}]$ , while keeping  $|v|$  sufficiently large and  $|\sin \vartheta(y)| \geq |\sin \vartheta(2y_0)|$ .  $\square$

**Proposition 6.2.2.** *If  $\alpha_1$  and  $\alpha_2$  are contact forms as described above, then*

$$ECH(M, \alpha_1) \simeq ECH(M, \alpha_2).$$

*Proof.* This is proved by finding a suitable contact extension of both  $(M, \alpha_1)$  and  $(M, \alpha_2)$  to a closed manifold  $M' = M \cup (S^1 \times D^2)$  and by applying Theorem 4.1.1. Here the meridian of  $V' = S^1 \times D^2$  is identified with a simple closed curve of slope  $\infty$  on  $\partial M$ . On the solid torus  $V'$ , we use cylindrical coordinates  $(\rho = 1 - y, \phi = 2\pi t, \theta)$ .

We claim that, given  $L > 0$ , there exists an extension of  $\alpha'$  to  $\alpha'(L)$  of the type given by Equation (2.4.1), such that all the closed orbits of  $R_{\alpha'(L)}$  have  $\alpha'(L)$ -action  $\geq L$ . This follows from Claim 6.2.1, which gives an extension to  $\{\frac{1}{2} \leq \rho \leq 1\} \subset V'$ ; a further extension to  $\{0 \leq \rho \leq \frac{1}{2}\}$  is straightforward. Here both extensions have no closed orbits of action  $\leq L$ . See Figure 4 for the function  $(f(\rho), g(\rho))$  on  $V'$ .

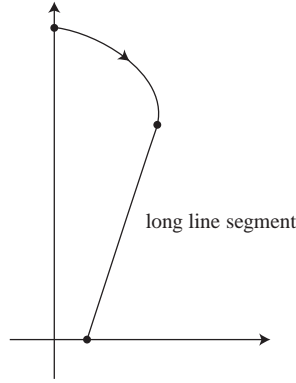


FIGURE 4. Trajectory of  $(f(\rho), g(\rho))$ . The arrow is in the direction of increasing  $\rho$ .

We now claim that the contact form  $\alpha'(L)$  satisfies:

$$ECC^{\leq L}(M, \alpha_k) = ECC^{\leq L}(M', \alpha_k \cup \alpha'(L)),$$

for  $k = 1, 2$ . Fix a particular  $k$ . Let  $\gamma$  be an orbit set of  $R_{\alpha_k}$  in  $M$  and  $\gamma'$  be an orbit set of  $R_{\alpha_k \cup \alpha'(L)}$  in  $M'$ . If  $\mathcal{A}_{\alpha_k}(\gamma) \leq L$ , then we claim that every holomorphic curve  $u(F)$  from  $\gamma$  to  $\gamma'$  is contained in  $\mathbb{R} \times M$ . First observe that  $\gamma'$  must be contained in  $\mathbb{R} \times M$  by action considerations. Next we use ideas from the blocking lemma (Lemma 5.7.1) to restrict  $u(F)$  to  $\mathbb{R} \times M$ : Let  $\{\rho = \rho_1\}$  be the torus where the slope of the Reeb vector field of  $\alpha'(L)$  is  $\infty$ . Then let  $\delta_{\rho_1+\varepsilon}$  be the intersection  $\pi_M(u(F)) \cap \{\rho = \rho_1 + \varepsilon\}$ , which we may take to be transverse. Since  $\gamma$  and  $\gamma'$  have no orbits in  $V'$ ,  $\delta_{\rho_1+\varepsilon}$  must have slope  $\infty$ . However, as  $\varepsilon \rightarrow 0$ ,  $\delta_{\rho_1+\varepsilon}$  approaches Reeb orbits of slope  $\infty$ , a contradiction unless  $\delta_{\rho_1+\varepsilon} = \emptyset$ . This implies that  $u(F)$  does not intersect  $\mathbb{R} \times \{\rho \leq \rho_0 + \varepsilon\}$ . Finally, if we write  $\delta_1 = \pi_M(u(F)) \cap \{\rho = 1\}$ ,

then  $[\delta_1] = 0 \in H_1(T)$  and the usual positivity of intersections argument shows that  $\delta_1 = \emptyset$  as well.

Next we claim that there is a constant  $c > 0$  — which depends on  $\alpha_1$  and  $\alpha_2$  but not on  $L > 0$  — so that there are cobordism maps  $\Phi$  and  $\overline{\Phi}$  which make the following diagram commute:

$$(6.2.1) \quad \begin{array}{ccc} ECH^{\leq L}(M, \alpha_1) & \xrightarrow{\sim} & ECH^{\leq L}(M', \alpha_1 \cup \alpha'(L)) \\ \overline{\Phi} \downarrow & & \downarrow \Phi \\ ECH^{\leq cL}(M, \alpha_2) & \xrightarrow{\sim} & ECH^{\leq cL}(M', \alpha_2 \cup \alpha'(L)) \end{array}$$

The constant  $c$  is an upper bound for  $|f|$ , where  $\phi^*(\alpha_2) = f\alpha_1$  for some diffeomorphism  $\phi$  of  $M$  which is isotopic to the identity and restricts to the identity on  $\partial M$ . The existence of the map  $\Phi$  follows from Theorem 4.1.1. The fact that  $\Phi$ , on the chain level, restricts to

$$\overline{\Phi} : ECC^{\leq L}(M, \alpha_1) \rightarrow ECC^{\leq cL}(M, \alpha_2),$$

follows by observing that the cobordism map used for  $\Phi$  is a symplectization on  $\mathbb{R} \times V'$  and by applying the argument in the previous paragraph.

Given a sequence  $L_j \rightarrow \infty$ , there exist constants  $0 < c_1 < c_2$  which do not depend on  $L_j$ , contact forms  $\alpha'_j = \alpha'(L_j)$ , and cobordism maps  $\overline{\Phi}_k, \Phi_{k,j}$ ,  $k = 1, 2, j = 1, 2, \dots$ , and which make the following diagram commute:

$$(6.2.2) \quad \begin{array}{ccc} ECH^{\leq L_j}(M, \alpha_1) & \xrightarrow{\sim} & ECH^{\leq L_j}(M', \alpha_1 \cup \alpha'_j) \\ \overline{\Phi}_1 \downarrow & & \downarrow \Phi_{1,j} \\ ECH^{\leq c_1 L_j}(M, \alpha_2) & \xrightarrow{\sim} & ECH^{\leq c_1 L_j}(M', \alpha_2 \cup \alpha'_j) \\ \overline{\Phi}_2 \downarrow & & \downarrow \Phi_{2,j} \\ ECH^{\leq c_2 L_j}(M, \alpha_1) & \xrightarrow{\sim} & ECH^{\leq c_2 L_j}(M', \alpha_1 \cup \alpha'_j) \end{array}$$

Moreover, by (iv) and (vi) of Theorem 4.1.1,  $\Phi_{2,j} \circ \Phi_{1,j}$  and  $\overline{\Phi}_2 \circ \overline{\Phi}_1$  are maps induced by the inclusion of chain complexes.

The left-hand side of Diagram (6.2.2), together with an analogous one for  $\overline{\Phi}_1 \circ \overline{\Phi}_2$ , imply that there is an isomorphism of direct limits

$$\lim_{j \rightarrow \infty} ECH^{\leq L_j}(M, \alpha_1) \xrightarrow{\sim} \lim_{j \rightarrow \infty} ECH^{\leq c_1 L_j}(M, \alpha_2),$$

which proves the claim.  $\square$

**6.3. The Morse-Bott point of view.** We would like to view  $ECH(M, \alpha)$  as a direct limit of ECH groups of nondegenerate contact forms as in Equation (5.5.1). Let us assume that  $\mathcal{N}$  is a negative Morse-Bott torus; the positive case is analogous, after replacing negative asymptotics with positive asymptotics. In order to accomplish this, we first pick  $L > 0$  and then slightly extend  $(M, \alpha)$  to  $(M_\varepsilon, \alpha)$  so that:

- $T \subset T^2 \times [-\varepsilon, \varepsilon]$  and  $M \cap (T^2 \times [-\varepsilon, \varepsilon]) = T^2 \times [-\varepsilon, 0]$ ;
- $T_s$  is foliated by Reeb orbits of  $\alpha$  for  $s \in [-\varepsilon, \varepsilon]$ ; and
- there are no Reeb orbits of  $\alpha$  on  $T_s$ ,  $s \in [-\varepsilon, \varepsilon] - \{0\}$ , which have action  $\leq L$ .

The situation is similar to that of Section 5.7.

We now consider the chain complex  $ECC^{\leq L}(M_\varepsilon, f_j\alpha, J_j)$ , where  $f_j$  and  $J_j$  satisfy (A2)–(A4) in Section 5.5. Here we take  $j$  to be sufficiently large so that  $g_j$  has support in  $\text{int}(T^2 \times [-\varepsilon/2, \varepsilon/2])$ . By Lemma 6.1.1, a sequence of  $J_j$ -holomorphic curves in  $\mathbb{R} \times \text{int}(M_\varepsilon)$  from  $\gamma$  to  $\gamma'$  cannot reach  $\mathbb{R} \times \partial M_\varepsilon$ , so  $\partial^2 = 0$  for each of  $ECC^{\leq L}(M_\varepsilon, f_j\alpha, J_j)$ . Now let  $u_j(F_j)$  be a  $J_j$ -holomorphic curve in  $\mathbb{R} \times \text{int}(M_\varepsilon)$  from  $\gamma$  to  $\gamma'$ , both of which have action  $\leq L$ . If we write  $\delta_{\pm\varepsilon}^j = \pi_M(u_j(F_j)) \cap T_{\pm\varepsilon}$  as before, then  $\delta_\varepsilon^j = \emptyset$ . Since the only orbits of action  $\leq L$  in  $T^2 \times [-\varepsilon, \varepsilon]$  are the orbits in  $\mathcal{N}$ ,  $\delta_{-\varepsilon}$  must have slope  $r$  by homological reasons. By the negativity of  $\mathcal{N}$ ,  $\delta_{-\varepsilon}$  must be oriented in the same direction as the Reeb orbits of  $\mathcal{N}$ . Hence an orbit in  $\mathcal{N}$  can only appear at the *negative* end of  $u_j(F_j)$ . This means the Morse-Bott ECH differential is completely described by Proposition 5.4.2.

The ECH group  $ECH(M, \alpha)$  can then be written as the direct limit of groups  $ECH^{\leq L}(M_\varepsilon, f_j\alpha, J_j)$ . As we discussed in Proposition 6.2.2, we complete  $M_\varepsilon$  to a closed manifold by attaching a solid torus and extending  $\alpha$  so that all the Reeb orbits away from  $M_\varepsilon$  have action  $> L$ .

## 7. VARIANTS OF ECH OF AN OPEN BOOK DECOMPOSITION

The goal of this section is to define the homology groups  $ECH(N, \partial N, \alpha)$  and  $\widehat{ECH}(N, \partial N, \alpha)$  which appear in the statement of Theorem 1.0.1. They are variants of  $ECH(N, \alpha)$  and in many ways can be viewed as ECH groups relative to the boundary of  $N$ , hence the notation. We will assume that the almost complex structure  $J$  on  $\mathbb{R} \times N$  is Morse-Bott regular, and suppress it from the notation.

Recall that  $\partial N$  is foliated by a *negative* Morse-Bott family  $\mathcal{N}$  of simple orbits of slope  $\infty$ . We assume without loss of generality that  $\alpha$  is nondegenerate away from  $\partial N$ , after a  $C^k$ -small perturbation for an arbitrary large

$k$ . We pick two orbits from  $\mathcal{N}$  and label them  $h$  and  $e$ . There is a perturbation of  $\alpha$  near  $\partial N$  which makes  $h$  hyperbolic and  $e$  elliptic. Moreover,  $I(h, e, Z) = 1$  if  $Z$  is the annulus between the two orbits on  $\partial N$ .

**7.1. Definition of  $ECH(N, \partial N, \alpha)$ .** Let  $\mathcal{P}$  be the set of Reeb orbits of  $\alpha$  in the interior of  $N$ . Let  $ECC_j^b(N, \alpha)$  be the chain group generated by orbit sets  $\Gamma$  constructed from  $\mathcal{P} \cup \{e\}$ , whose homology class  $[\Gamma]$  intersects the page  $S \times \{t\}$  exactly  $j$  times. The Morse-Bott ECH differential counts  $I = 1$  Morse-Bott buildings between orbits sets constructed from  $\mathcal{P} \cup \{e\}$ . Recall that  $I = 1$  Morse-Bott buildings are nice by Proposition 5.4.2. By construction,  $ECC_j^b(N, \alpha)$  is a direct summand of  $ECC^b(N, \alpha)$  and the differential for  $ECC_j^b(N, \alpha)$  is the restriction of the differential for  $ECC^b(N, \alpha)$ . Since  $\mathcal{N}$  is a *negative* Morse-Bott family and  $\mathcal{N}$  is on the boundary of  $N$ , an orbit of  $\mathcal{N}$  can only appear at the negative end.

There are inclusions of chain complexes:

$$\begin{aligned} ECC_j^b(N, \alpha) &\rightarrow ECC_{j+1}^b(N, \alpha), \\ \Gamma &\mapsto e\Gamma, \end{aligned}$$

where we are using multiplicative notation for orbit sets. The homology of the chain complex  $ECC_j^b(N, \alpha)$  will be written as  $ECH_j^b(N, \alpha)$ . We then define

$$ECH(N, \partial N, \alpha) = \lim_{j \rightarrow \infty} ECH_j^b(N, \alpha).$$

*Remark 7.1.1.* Another way of interpreting  $ECH(N, \partial N, \alpha)$  is as the homology of the chain complex obtained by taking the quotient of the chain complex  $ECC^b(N, \alpha)$  by the subgroup generated by  $\{e\Gamma - \Gamma\}$ , where  $\Gamma$  is any orbit set constructed from  $\mathcal{P} \cup \{e\}$ .

**7.2. Definition of  $\widehat{ECH}(N, \partial N, \alpha)$ .** We write  $ECC_j(N, \alpha)$  for the chain group generated by orbit sets  $\Gamma$  constructed from  $\mathcal{P} \cup \{e, h\}$ , such that the homology class  $[\Gamma]$  intersects the page  $S \times \{t\}$  exactly  $j$  times. By construction,  $ECC_j(N, \alpha)$  is a direct summand of  $ECC(N, \alpha)$  and the differential for  $ECC_j(N, \alpha)$  is the restriction of the differential for  $ECC(N, \alpha)$ . There are inclusions of chain complexes:

$$\begin{aligned} ECC_j(N, \alpha) &\rightarrow ECC_{j+1}(N, \alpha), \\ \Gamma &\mapsto e\Gamma, \end{aligned}$$

as before. Writing  $ECH_j(N, \alpha)$  for the homology of the chain complex  $ECC_j(N, \alpha)$ , we define

$$\widehat{ECH}(N, \partial N, \alpha) = \lim_{j \rightarrow \infty} ECH_j(N, \alpha).$$

*Remark 7.2.1.* Another way of interpreting  $\widehat{ECH}(N, \partial N, \alpha)$  is as the homology of the chain complex obtained by taking the quotient of the usual ECH chain complex  $ECC(N, \alpha)$  by the subgroup generated by  $\{e\Gamma - \Gamma\}$ , where  $\Gamma$  is any orbit set constructed from  $\mathcal{P} \cup \{e, h\}$ .

**7.3. Intuitive idea behind Theorem 1.0.1.** We briefly explain the intuitive idea behind Theorem 1.0.1. Suppose for the moment that the contact form  $\alpha$  on  $M$ , near the binding, is given by Example 1 in Section 2.4.2. In other words, apart from the binding, the concentric tori  $\rho = \text{const}$  are foliated by Reeb orbits of irrational slope  $\frac{1}{\delta}$ . We would like to take the limit as  $\delta \rightarrow 0$ ; in the limit  $\partial(S^1 \times D^2)$  is foliated by Reeb orbits of slope  $\infty$ . There should be a one-to-one correspondence, modulo  $\mathbb{R}$ -translations, between holomorphic curves  $u$  in  $\mathbb{R} \times M$  of ECH index 1 which intersect the binding  $k$  times, and holomorphic curves  $u'$  in  $\mathbb{R} \times N$  of ECH index 1 which have negative ends at  $e$  with total multiplicity  $k$ . Also, as we take  $\delta \rightarrow 0$ , the Conley-Zehnder index of the binding, measured with respect to the longitudinal framing on  $S^1 \times D^2$ , i.e., the framing given by  $(S \times \{pt\}) \cap \partial(S^1 \times D^2)$ , goes to  $\infty$ . This suggests that we should be able to effectively ignore the binding if we could take the limit.

The actual proof — at least the one we could find — is considerably more complicated, and uses three ingredients: (i) the calculation of ECH on the solid torus, for specific contact forms, (ii) some understanding of holomorphic curves on the region  $\mathbb{R} \times T^2 \times [1, 2]$ , and (iii) spectral sequences. The proofs will be given in Sections 8 and 9.

## 8. ECH OF THE SOLID TORUS

In this section we calculate ECH of the solid torus with certain boundary conditions. We will be using the same slope convention for embedded curves in  $\partial V$  as in Section 2.

The following is the main result of this section:

**Theorem 8.0.1.** *Let  $\alpha$  be a contact form on  $V$  such that  $\partial V$  is foliated by a positive Morse-Bott family of closed Reeb orbits of slope infinity. Then  $ECH(\text{int}(V), \alpha) \simeq \mathbb{F}$ , generated by  $\emptyset$ .*

We have two options: (i) Do an explicit calculation modeled on the work of Hutchings-Sullivan [HS1, HS2], where they calculate ECH of  $T^2 \times [0, 1]$  and  $T^3$  for certain contact forms. This is enabled by the list of holomorphic curves on  $\mathbb{R} \times S^1 \times D^2$  which is provided by Taubes [T3]. (ii) Apply a direct limit argument based on Theorem 4.1.1. Although (i) can be done without too much difficulty, we choose (ii) since it involves less work.

*Proof.* Let  $\alpha$  be the contact form  $\alpha_\delta$ , defined in Section 2.4.3. The proof is carried out in several steps.



**Step 1.** Let  $r > 0$  be an irrational number. Pick a contact form  $\alpha_r$  on  $V \simeq S^1 \times D^2$  so that the following hold:

- the boundary  $\partial V$  and all the concentric tori  $T_\rho$ ,  $\rho \in (0, 1]$ , are foliated by Reeb orbits of irrational slope  $r$ ;
- the contact structure  $\ker \alpha_r$  is transverse to all the fibers  $S^1 \times \{pt\}$ .

There is only one simple closed orbit, namely the core  $e = S^1 \times \{0\}$ . The orbit  $e$  is elliptic and all its multiple covers  $e^n$  are nondegenerate due to the irrationality of  $r$ . Note that  $[e^n] = n[S^1] \in H_1(V)$ .

Let  $\xi$  be a contact structure on  $V$  which is transverse to all the fibers  $S^1 \times \{pt\}$ . Then there is a trivialization  $\tau(D^2)$  of  $\xi$  which is obtained by pulling back a trivialization of  $TD^2$  via the projection  $S^1 \times D^2 \rightarrow D^2$  onto the second factor. The orbit  $e^n$  of the Reeb vector field  $R_{\alpha_r}$  has Conley-Zehnder index  $\mu_{\tau(D^2)}(e^n; \alpha_r) = 2\lfloor nr \rfloor + 1$ .

Next, in order to use direct limits, we lift the relative grading given by the ECH index  $I$  to an absolute grading which is as much as possible independent of the contact form. For each  $n > 0$ , let  $\alpha_r^n = f_{r,n}(\rho)\alpha_r$  be a reference contact form on  $V$  such that the following hold:

- (1)  $f_{r,n}(\rho) = 1$  near  $\rho = 1$ ;
- (2) the core  $e$  is an orbit of  $R_{\alpha_r^n}$  and  $\mu_{\tau(D^2)}(e^n; \alpha_r^n) = 1$ ;
- (3) the contact forms  $\alpha_r^n$  all agree on a neighborhood of  $e$ , where  $n$  is fixed and  $r$  is varying.

By interpolating from  $f_{r,n}$  to 1, we can construct an exact symplectic cobordism  $(\mathbb{R} \times V, \Omega_{r,n})$  which has the symplectization of  $\alpha_r^n = f_{r,n}\alpha_r$  at the positive end and the symplectization of  $\alpha_r$  at the negative end. The definition of the ECH index for cobordisms was given in [Hu2, Definition 4.3], and is analogous to Equation (3.4.2). We first set  $I(e^n; \alpha_r^n) = 1$  to be the reference normalization. We then compute:

$$(8.0.1) \quad I(e^n; \alpha_r^n) - I(e^n; \alpha_r) = \mu_{\tau(D^2)}(e^n; \alpha_r^n) - \mu_{\tau(D^2)}(e^n; \alpha_r).$$

This is done by taking the surface  $Z$  from  $e^n$  to  $e^n$  in the symplectization to be  $n$  copies of the trivial cylinder from  $e$  to  $e$ . The intersection pairing and the Chern class term for  $Z$  with respect to  $\tau(D^2)$  can easily be computed to be zero. We then observe that, since  $H_2(V) = 0$ , the left-hand side of Equation (8.0.1) does not depend on the choice of  $Z$ . Hence

$$I(e^n; \alpha_r) = \mu_{\tau(D^2)}(e^n; \alpha_r) = 2\lfloor nr \rfloor + 1.$$

When  $n = 0$ , then  $e^n = \emptyset$ , which we normalize to have  $I(\emptyset; \alpha_r) = 0$ .

Summarizing, we have:

**Claim 8.0.2.** *ECH( $V, \alpha_r, n[S^1]$ ) is generated by  $e^n$  for integers  $n \geq 0$ , and is zero for  $n < 0$ . Moreover,*

$$I(e^n; \alpha_r) = \mu_{\tau(D^2)}(e^n; \alpha_r) = 2\lfloor nr \rfloor + 1$$

for  $n \geq 1$  and  $I(\emptyset; \alpha_r) = 0$ .

A similar calculation shows that for a pair of irrational numbers  $r, r' > 0$ , the normalizations  $I(e^n; \alpha_r^n) = 1$  and  $I(e^n; \alpha_{r'}^n) = 1$  are consistent.

**Step 2.** Let  $r_i \rightarrow \infty, i \in \mathbb{Z}^{\geq 0}$ , be a sequence of irrational numbers. By the construction of the contact form  $\alpha$  on  $V$ , there is a unique concentric solid torus  $V_i \subset V$  whose boundary is foliated by Reeb orbits of  $\alpha$  of slope  $r_i$ , provided  $r_i$  is sufficiently large.

Let  $\alpha_1 = \alpha|_{V_i}$  and  $\alpha_2 = (\phi_{r_i})_* \alpha_{r_i}$  be contact forms on  $V_i$ , where  $\phi_{r_i} : V \xrightarrow{\sim} V_i$  is a deformation retraction of  $V$  onto  $V_i$ . The contact form  $\alpha_{r_i}$  can be chosen such that the Reeb vector fields  $R_{\alpha_1}$  and  $R_{\alpha_2}$  agree on  $\partial V_i$  and the characteristic foliations of  $\xi_1 = \ker \alpha_1$  and  $\xi_2 = \ker \alpha_2$  agree on  $\partial V_i$ . In view of Proposition 6.2.2, we can reinterpret Claim 8.0.2 as follows:

**Corollary 8.0.3.**  *$ECH(V_i, \alpha|_{V_i}, n[S^1]) \simeq \mathbb{F}$  for integers  $n \geq 0$  and is zero for  $n < 0$ . The generator for  $n \geq 1$  has ECH index  $I = 2\lfloor nr_i \rfloor + 1$  and the generator  $\emptyset$  for  $n = 0$  has  $I = 0$ .*

### Step 3.

**Claim 8.0.4.** *The inclusions  $V_i \subset V_{i+1}$  induce inclusions of chain complexes*

$$(8.0.2) \quad ECC(V_i, \alpha|_{V_i}) \rightarrow ECC(V_{i+1}, \alpha|_{V_{i+1}}).$$

*Proof.* Let  $\gamma$  (resp.  $\gamma'$ ) be an orbit set whose orbits are contained in  $V_i$  (resp.  $V_{i+1}$ ), and let  $u(F)$  be a holomorphic curve in  $\mathbb{R} \times V_{i+1}$  from  $\gamma$  to  $\gamma'$ .

We first prove that all the orbits of  $\gamma'$  must be contained in  $V_i$ . Arguing by contradiction, suppose  $\gamma' = \gamma'_{in} \gamma'_{out}$ , where the orbits of  $\gamma'_{in}$  are in  $V_i$  and the orbits of  $\gamma'_{out} \neq \emptyset$  are in  $V_{i+1} - V_i$ . Let  $T_{\rho_0} = \{\rho = \rho_0\}$  be the torus which contains an orbit of  $\gamma'_{out}$  with the largest  $\rho$ -value. (Note that  $\rho$  increases towards the boundary of  $V$ .) Also let  $\gamma'_{\rho_0}$  be the orbit subset of  $\gamma'_{out}$  consisting of orbits on  $T_{\rho_0}$ . Writing  $\pi_V$  for the projection  $\mathbb{R} \times V \rightarrow V$ , we consider the intersection  $\delta_\varepsilon = \pi_V(u(F)) \cap T_{\rho_0 - \varepsilon}$  for  $\varepsilon > 0$ . By Lemma 5.6.1,  $\delta_\varepsilon$  is transverse to the Reeb orbits of  $T_{\rho_0 - \varepsilon}$  for all  $\varepsilon > 0$  sufficiently small; in particular,  $\delta_\varepsilon$  is immersed. Moreover,  $[\delta_\varepsilon] = [\gamma'_{\rho_0}]$  on

$$H_1(T^2 \times [\rho_0 - \varepsilon, \rho_0]) \simeq H_1(T_{\rho_0}).$$

As  $\varepsilon \rightarrow 0$ ,  $\delta_\varepsilon$  must approach  $\gamma'_{\rho_0}$ . This is incoming behavior at a positive end of a symplectization, contradicting the fact that  $\gamma'_{\rho_0}$  must be at the negative end.

Once we know that  $\gamma$  and  $\gamma'$  are both contained in  $V_i$ , the argument from Lemma 6.1.1 shows that  $u(F)$  is contained in  $\mathbb{R} \times V_i$ .  $\square$

Hence we can write

$$(8.0.3) \quad ECH(int(V), \alpha) = \lim_{i \rightarrow \infty} ECH(V_i, \alpha|_{V_i}).$$

**Step 4.** We now use Corollary 8.0.3 and the direct limit given by Equation (8.0.3). For each  $r_i$ ,  $ECH(V_i, \alpha|_{V_i}, n[S^1]) \simeq \mathbb{F}$ , whose generator has ECH index  $I = 2\lfloor nr_i \rfloor + 1$  if  $n \geq 1$ . If we take  $r_{i+1} \gg r_i$ , then  $2\lfloor nr_{i+1} \rfloor + 1 > 2\lfloor nr_i \rfloor + 1$  for all  $i > 0$ . Hence the map given by Equation (8.0.2) is the zero map for homology classes  $n[S^1]$  with  $n > 0$ . On the other hand,  $\emptyset$  is the only generator of  $ECC(V_i, \alpha|_{V_i}, 0)$ , and survives passage to the direct limit. It follows that  $ECH(int(V), \alpha) = \mathbb{F}$ , generated by  $\emptyset$ .

This completes the proof of Theorem 8.0.1.  $\square$

We now prove a corollary of Theorem 8.0.1 for the variants of the ECH group  $ECH(int(V), \alpha)$  which were defined in Section 6.1. Recall that  $\partial V$  is foliated by a *positive* Morse-Bott family. To remember this positive sign, and for consistency with the notation used in the following section, we will denote the elliptic and hyperbolic orbits coming from a nondegenerate perturbation of  $\partial V$  by  $e'$  and  $h'$ .

**Corollary 8.0.5.** *Let  $\alpha$  be a contact form on  $V$  such that  $\partial V$  is foliated by a positive Morse-Bott family of closed Reeb orbits of slope  $\infty$ . Then the following hold:*

- (1)  $ECH^\sharp(V, \alpha) = 0$ .
- (2)  $ECH(V, \alpha) = 0$ .
- (3)  $ECH^b(V, \alpha) = \mathbb{F}[e']$ , where  $\mathbb{F}[e']$  is the polynomial ring generated by  $e'$  over  $\mathbb{F}$ .

*Proof.* (1) Consider the filtration  $\mathcal{F}_1$  on  $ECC^\sharp(V, \alpha, n[S^1])$  which is defined on the generators as follows: Given an orbit set  $(h')^k \gamma$ , where  $\gamma$  does not have any  $h'$ -terms, we set

$$\mathcal{F}_1((h')^k \gamma) = k.$$

The differential is filtration-nonincreasing. If  $n > 0$ , then the  $E^1$ -term of the spectral sequence<sup>1</sup> computes  $ECH(int(V), \alpha, n[S^1])$  in both filtration levels 0 and 1. By Theorem 8.0.1,

$$ECH(int(V), \alpha, n[S^1]) = 0$$

for  $n > 0$ , and the spectral sequence converges to 0.

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<sup>1</sup>Of course a spectral sequence for a filtered complex of length 2 is nothing but the relative exact sequence and the use of spectral sequences in this context may appear silly. We chose to do so in order to present the cases when we filter by the multiplicity of  $h'$  and when we filter by the multiplicity of  $e'$  in a unified way.

What we have left is  $n = 0$ , which is generated by  $h'$  and  $\emptyset$ . We now appeal to Taubes [T3] or Wendl [We2, p.43], where it is shown that there is an  $\mathbb{R} \times S^1$ -family of holomorphic planes in  $\mathbb{R} \times V$  which are asymptotic to the Morse-Bott family on  $\partial V$ . This gives rise to a finite energy foliation which constrains holomorphic curves which have an orbit of  $\partial V$  at the positive end (see Section 9.2 for an almost identical argument). We obtain  $\partial h' = \emptyset$  by counting Morse-Bott buildings of ECH index one. This proves the vanishing of  $ECH^\sharp(V, \alpha, 0)$ .

(2) We now define a filtration  $\mathcal{F}_2$  on  $ECC(V, \alpha)$  as follows: Given an orbit set  $(e')^m \gamma$ , where  $\gamma$  does not have any  $e'$ -terms, we set

$$\mathcal{F}_2((e')^m \gamma) = m.$$

The  $E^1$ -term of the spectral sequence computes  $ECH^\sharp(V, \alpha)$  in each filtration level. By (1),  $ECH^\sharp(V, \alpha) = 0$ , and the spectral sequence converges to 0.

(3) Let  $\mathcal{F}_3$  be a filtration on  $ECC^b(V, \alpha)$  as follows: Given an orbit set  $(e')^m \gamma$ , where  $\gamma$  does not have any  $e'$ -terms, let  $\mathcal{F}_3((e')^m \gamma) = m$ . The  $E^1$ -term of the spectral sequence for  $\mathcal{F}_3$  computes

$$\bigoplus_{m=0}^{\infty} ECH(int(V), \alpha) \cdot (e')^m.$$

Since  $ECH(int(V), \alpha) \simeq \mathbb{F}\{\emptyset\}$  by Theorem 8.0.1, the  $E^1$ -term of the spectral sequence is  $\mathbb{F}[e']$ . All higher differentials vanish for degree reasons; therefore  $E^1 = E^\infty$  is the graded group of the induced filtration on  $ECH^b(V, \alpha)$ . Since  $\mathbb{F}[e'] \subset ECC^b(V, \alpha)$  is a subset of cycles, the corollary follows.  $\square$

## 9. PROOF OF THEOREM 1.0.1

In this section we prove Theorem 1.0.1. The proof was greatly influenced by Michael Hutchings, who encouraged us to look for an appropriate filtration.

**9.1. Proof of Theorem 1.0.1(1).** We start with a description of the Reeb orbits of  $R_{\alpha_\delta}$  on  $M$ , which was constructed in Section 2. Recall that  $M$  can be decomposed as  $N \cup (T^2 \times [1, 2]) \cup V$ .

### Description of Reeb orbits of $R_{\alpha_\delta}$ .

(1) The boundary of  $V$  is foliated by a Morse-Bott family  $\mathcal{N}_2$  of slope  $\infty$ . Let  $h'$  and  $e'$  be the orbits in  $\mathcal{N}_2$  which become hyperbolic and elliptic after perturbation. (The orbits in  $int(V)$  are not so important in view of the calculations done in Section 8.)

(2) The Reeb orbits in the interior of  $T^2 \times [1, 2]$  come in Morse-Bott families of large negative slope which depend on the choice of  $\alpha_\delta$ . Such Reeb orbits have arbitrarily large action, and can be discarded by a direct limit argument which sends  $\delta \rightarrow 0$ .

(3) The boundary of  $N$  is foliated by a Morse-Bott family  $\mathcal{N}_1$  of slope  $\infty$ . Let  $h$  and  $e$  be the orbits in  $\mathcal{N}_1$  which become hyperbolic and elliptic after perturbation.

**Direct limit.** We can take a sequence of positive numbers  $\delta_i \rightarrow 0$ ,  $i = 0, 1, \dots$ , in the construction of  $\alpha_{\delta_i}$  in Section 2.3.3. The sequence  $\alpha_{\delta_i}$  is commensurate to a fixed  $\alpha_{\delta_0}$  on  $T^2 \times [1, 2]$  via diffeomorphisms  $\phi_i$  of  $T^2 \times [1, 2]$  which restrict to the identity on  $\partial(T^2 \times [1, 2])$ . See Definition 4.2.2. By Corollary 4.2.3, given a sequence  $L_i \rightarrow \infty$  with  $L_{i+1} > \frac{1}{c^2} L_i$ , we have:

$$ECH(M) = \lim_{i \rightarrow \infty} ECH^{\leq L_i}(M, \alpha_{\delta_i}).$$

Moreover, in the direct limit, we can first choose  $L_i$  and then choose  $\delta_i$  so that  $ECC^{\leq L_i}(M, \alpha_{\delta_i})$  does not contain any Reeb orbits in  $\text{int}(T^2 \times [1, 2])$ . Since the Reeb orbits in  $V \cup N$  and holomorphic curves on  $\mathbb{R} \times (V \cup N)$  do not depend on  $\delta$ , we can effectively discard the orbits in  $\text{int}(T^2 \times [1, 2])$ .

In view of the above discussion, we may replace the chain complex  $(ECC(M, \alpha), \partial)$  by its subcomplex  $(ECC(V, \alpha) \otimes ECC(N, \alpha), \partial)$ . Note that the tensor product is only as vector spaces, and not as chain complexes. The generators of  $ECC(V, \alpha) \otimes ECC(N, \alpha)$  will be written as  $\gamma \otimes \Gamma$ , where  $\gamma$  is an orbit set in  $V$  and  $\Gamma$  is an orbit set in  $N$ . Also  $\partial$  denotes both the differential on  $ECC(M, \alpha)$  and its restriction to  $ECC(V, \alpha) \otimes ECC(N, \alpha)$ .

**The first filtration.** Choose an identification  $\eta : H_1(V; \mathbb{Z}) \xrightarrow{\sim} \mathbb{Z}$  so that the homology class of the binding is mapped to 1. Then we define the filtration  $\mathcal{F} : ECC(M) \rightarrow \mathbb{Z}^{\geq 0}$  as follows:

$$\mathcal{F} \left( \sum_i \gamma_i \otimes \Gamma_i \right) = \max_i \eta([\gamma_i]).$$

Note that the filtration is bounded below.

**Lemma 9.1.1.** *The differential  $\partial$  for  $ECC(V, \alpha) \otimes ECC(N, \alpha)$  respects the filtration  $\mathcal{F}$ .*

*Proof.* Let  $u : (F, j) \rightarrow (\mathbb{R} \times M, J)$  be a holomorphic curve which is asymptotic to  $\gamma \otimes \Gamma$  at the positive end and to  $\gamma' \otimes \Gamma'$  at the negative end. We prove that  $\mathcal{F}(\gamma \otimes \Gamma) \geq \mathcal{F}(\gamma' \otimes \Gamma')$ .

Let  $\varepsilon \geq 0$  be a small constant. Writing  $V_\varepsilon = V \cup (T^2 \times [2 - \varepsilon, 2])$ , we consider the restriction of  $u$  to  $F'_\varepsilon = u^{-1}(\mathbb{R} \times V_\varepsilon)$ . Let  $\pi_M : \mathbb{R} \times M \rightarrow M$  be

the projection onto the second factor. We view the boundary of  $\pi_M(u(F'_\varepsilon))$  as  $\gamma - \gamma' + \delta_\varepsilon$ , where  $\delta_\varepsilon = \pi_M(u(F'_\varepsilon)) \cap T_{2-\varepsilon}$ .

We claim that  $\eta([\delta_\varepsilon]) \leq 0$  for  $\varepsilon = 0$  or for all sufficiently small  $\varepsilon > 0$ . First suppose that  $\pi(u(F))$  intersects  $T_2 = \partial V$  transversely and that  $\gamma$  and  $\gamma'$  do not involve Reeb orbits of  $T_2 = \partial V$ . (The first condition is equivalent to  $\pi(u(F)) \cap \partial V$  intersecting the Reeb vector field  $R$  transversely.) The claim then follows from the positivity of intersections in dimension four (i.e.,  $u(F)$  intersects each trivial cylinder over a Reeb orbit positively) and the fact that the torus  $T_2 = \partial V$  is foliated by Reeb orbits of slope  $\infty$ .

In the general case, for all sufficiently small  $\varepsilon > 0$ ,  $\delta_\varepsilon$  has no singular points by the finiteness of  $\text{wind}_\pi(u(F))$ . Hence, for sufficiently small  $\varepsilon > 0$ , the  $\delta_\varepsilon$  are isotopic to one another and are transverse to  $R$  on  $T_{2-\varepsilon}$ . It follows that each connected component  $\delta'$  of  $\delta_\varepsilon$  satisfies  $\eta([\delta']) \leq 0$ .

Finally, since

$$\eta([\gamma]) + \eta([\delta_\varepsilon]) = \eta([\gamma']),$$

we obtain  $\mathcal{F}(\gamma \otimes \Gamma) \geq \mathcal{F}(\gamma' \otimes \Gamma')$ . This completes the proof.  $\square$

*Remark 9.1.2.* It is possible for a connected component  $\delta'$  of  $\delta_\varepsilon$  to satisfy  $\eta([\delta']) = 0$ , where  $\varepsilon > 0$  is sufficiently small. For each such component  $\delta'$ , the holomorphic curve  $F$  has a positive asymptotic end which covers a Reeb orbit in the Morse-Bott family on  $T_2$ . This is justified in the context of the perturbation of the Morse-Bott family by the blocking lemma (Lemma 5.7.1).

The filtration  $\mathcal{F}$  induces a spectral sequence  $E^r(\mathcal{F})$ . The term  $E^0(\mathcal{F})$ , corresponding to the graded complex associated to  $\mathcal{F}$ , can be identified (as a vector space) with  $ECC(V, \alpha) \otimes ECC(N, \alpha)$ , since  $ECC(V, \alpha) \otimes ECC(N, \alpha)$  is freely generated by homogeneous elements. The differential  $\partial_0$  on  $E^0(\mathcal{F}) \cong ECC(V, \alpha) \otimes ECC(N, \alpha)$  is the filtration-preserving component of  $\partial$ . Every sheet  $E^r(\mathcal{F})$  has a grading coming from  $\mathcal{F}$ , and the component in degree  $p$  of  $E^r(\mathcal{F})$  will be denoted by  $E_p^r(\mathcal{F})$ .

**Theorem 9.1.3.** *The  $E^1$ -term of the spectral sequence for the filtration  $\mathcal{F}$  is*

$$E_p^1(\mathcal{F}) = \begin{cases} ECH(N, \partial N), & \text{if } p = 0; \\ 0, & \text{if } p > 0. \end{cases}$$

Assuming Theorem 9.1.3 (which will be proven in Section 9.3) for the moment, we have the following:

*Proof of Theorem 1.0.1(1).* The proof of Theorem 1.0.1(1) is a trivial consequence of a standard fact about spectral sequences. Since  $E^1(\mathcal{F})$  is concentrated in a single degree, all higher differentials vanish and the spectral sequence collapses at  $E^1(\mathcal{F})$ . Moreover the extension problem is trivial; hence  $ECH(M) \cong E^1(\mathcal{F}) \cong ECH(N, \partial N)$ .  $\square$

We observe that the isomorphism is canonical, since  $E_0^1(\mathcal{F})$  is the homology of the lowest filtration level  $\mathcal{F}_0(ECC(V, \alpha) \otimes ECC(N, \alpha))$ , which is a subcomplex of  $ECC(V, \alpha) \otimes ECC(N, \alpha)$ , and the map in the theorem is induced by the inclusion  $\mathcal{F}_0(ECC(V, \alpha) \otimes ECC(N, \alpha)) \hookrightarrow ECC(V, \alpha) \otimes ECC(N, \alpha)$ .

**9.2. Holomorphic curves on  $\mathbb{R} \times T^2 \times [1, 2]$ .** In this subsection we discuss holomorphic curves on  $\mathbb{R} \times T^2 \times [1, 2]$ . Our aim is to obtain a detailed description of the differential  $\partial_0$  (i.e., Lemma 9.2.3). The key ingredient is a finite energy foliation of  $\mathbb{R} \times \text{int}(\mathbb{R} \times [1, 2])$  which is furnished by Wendl [We].

We can take the almost complex structure  $J$  on  $\mathbb{R} \times T^2 \times [1, 2]$  with coordinates  $(s, \theta, t, y)$  so that the following holds:

- $J$  is invariant in the  $s, \theta, t$ -directions;
- $J$  sends  $\partial_y \in \xi$  to the tangent space to  $T_y = T^2 \times \{y\}$ .

**9.2.1. The finite energy foliation.** Let  $\mathcal{N}_i, i = 1, 2$ , be the Morse-Bott family of simple Reeb orbits corresponding to  $T_i$ . The following is proved in Wendl [We, Section 4.2]:

**Lemma 9.2.1** (Wendl). *There is an  $(\mathbb{R} \times \mathbb{R}/\mathbb{Z})$ -family of holomorphic cylinders  $Z_{s,\theta}$ ,  $(s, \theta) \in \mathbb{R} \times \mathbb{R}/\mathbb{Z}$ , on  $\mathbb{R} \times T^2 \times [1, 2]$  which foliate  $\mathbb{R} \times \text{int}(T^2 \times [1, 2])$  and project to cylinders  $\theta = \text{const}$  on  $\text{int}(T^2 \times [1, 2])$ . Each cylinder  $Z_{s,\theta}$  is positively asymptotic to a Reeb orbit in  $\mathcal{N}_2$  and negatively asymptotic to a Reeb orbit in  $\mathcal{N}_1$ .*

Here  $Z_{s,\theta}$  is the image of some holomorphic map  $u_{s,\theta} : \mathbb{R} \times S^1 \rightarrow \mathbb{R} \times (T^2 \times [1, 2])$ .

*Proof.* We simply observe that our  $J$  satisfies Equation (34) of [We]. Wendl solves a dimensionally reduced Cauchy-Riemann equation (Equations (37a) and (37b) of [We]), which projects to cylinders  $\theta = \text{const}$  on  $\text{int}(T^2 \times [1, 2])$ .  $\square$

**9.2.2. Restricting curves in  $\mathbb{R} \times T^2 \times [1, 2]$ .** The finite energy foliation  $\{Z_{s,\theta}\}$  strongly constrains the holomorphic curves that can enter  $\mathbb{R} \times T^2 \times [1, 2]$ , as summarized in Lemma 9.2.2. Recall that  $N' = N \cup (T^2 \times [1, 2])$ .

**Lemma 9.2.2.** *Let  $u : (F, j) \rightarrow \mathbb{R} \times N'$  be a finite energy holomorphic map, where  $F$  is a closed Riemann surface with finitely many punctures. If there is an end of  $u$  which is asymptotic to some orbit in  $T_2$  but no ends which are asymptotic to orbits in the interior of  $T^2 \times [1, 2]$ , then  $u(F)$  must equal  $Z_{s,\theta}$  for some  $(s, \theta)$ .*



*Proof.* This is a consequence of the positivity of intersections and the fact that  $\{Z_{s,\theta}\}$  is a finite energy foliation of  $\mathbb{R} \times \text{int}(T^2 \times [1, 2])$ .

First observe that  $u(F)$  is strictly contained in  $\mathbb{R} \times T^2 \times [1, 2]$  if there is an end of  $u$  which is asymptotic to an orbit in  $T_2$  but no ends which are asymptotic to orbits in  $\text{int}(T^2 \times [1, 2])$ . This follows from an argument similar to that of the blocking lemma, i.e., by considering  $\pi(u(F)) \cap T_{1-\varepsilon}$  for  $\varepsilon > 0$  small. Here  $\pi : \mathbb{R} \times N' \rightarrow N'$  be the natural projection. The negative asymptotic ends of  $u(F)$  are Reeb orbits of  $T_1$ .

Suppose that  $\pi(u(F)) \not\subset \pi(Z_{s,\theta})$  for any  $s, \theta$ . Then there exists  $\theta_0$  so that  $K = \pi(u(F)) \cap \pi(Z_{s,\theta_0})$  is a (nonempty) immersed closed 1-dimensional submanifold of the annulus  $\{\theta = \theta_0\} \subset T^2 \times [1, 2]$ . In particular, there exists  $s_0 \in \mathbb{R}$  so that  $u(F)$  and  $Z_{s_0,\theta_0}$  intersect nontrivially. By the positivity of intersections, the intersection  $u(F) \cap Z_{s_0,\theta_0}$  is positive. Since we can take  $F_{s_0,\theta_0}$  so that the Reeb orbits at its ends are different from those of  $u(F)$ , the intersection  $u(F) \cap Z_{s,\theta_0}$  must be positive for all  $s \in \mathbb{R}$ . Now, the preimage  $\pi^{-1}(K) \cap u(F)$  of  $K$  in  $u(F)$  is compact. Hence if we take  $s$  to be arbitrarily large, then  $Z_{s,\theta_0}$  cannot intersect  $u(F)$ , a contradiction.

On the other hand, suppose  $\pi(u(F)) \subset \{\theta = \theta_0\} \subset T^2 \times [1, 2]$  for some  $\theta_0$ . Then  $\mathbb{R} \times \{\theta = \theta_0\}$  is a 3-manifold foliated by  $Z_{s,\theta_0}$ ,  $s \in \mathbb{R}$ . If  $u(F) \neq Z_{s,\theta_0}$  for any  $s$ , then  $u(F)$  must intersect some  $Z_{s,\theta_0}$  along a subset of dimension 1, which is too large for intersections between holomorphic curves, a contradiction.  $\square$

**9.2.3. Automatic transversality.** The finite energy foliation  $\{Z_{s,\theta}\}$  not only is used to restrict curves passing through  $\mathbb{R} \times T^2 \times [1, 2]$ , but also contributes to the ECH differential.

We first discuss the Fredholm index of a Morse-Bott building in our setting. Let  $h, e \in \mathcal{N}_1$  and  $h', e' \in \mathcal{N}_2$  be orbits which are to be perturbed into hyperbolic and elliptic orbits. Let  $\tilde{u}_1$  be a Morse-Bott building from  $e'$  to  $h$ , which consists of a gradient trajectory in  $\mathcal{N}_2$  and some  $Z_{s,\theta}$ . If we define the Conley-Zehnder indices as in Definition 5.4.1, then the Fredholm index  $\text{ind}(\tilde{u}_1)$  equals one. Similarly, if  $\tilde{u}_2$  is a Morse-Bott building from  $h'$  to  $e$  which consists of some  $Z_{s,\theta}$  and a gradient trajectory in  $\mathcal{N}_1$ , then  $\text{ind}(\tilde{u}_2) = 1$ .

In order to count curves in the finite energy foliation, we need to prove the regularity of the Morse-Bott family  $Z_{s,\theta}$  of holomorphic cylinders. We invoke the automatic transversality result for Morse-Bott families due to Wendl [We2, We3]. Here we will rephrase the result to suit our context: The Morse-Bott building  $\tilde{u}_i$  is regular if

$$\text{ind}(\tilde{u}_i) \geq 2g(\tilde{u}_i) + \#\Gamma_0,$$

where  $\#\Gamma_0$  is the number of ends which are hyperbolic orbits with positive eigenvalues and  $g(\tilde{u}_i)$  is the genus of  $\tilde{u}_i$ . (Wendl's formula has another parity term, but it cancels because we have two Morse-Bott families  $T_2$  and  $T_1$ , and they have canceling parities.) Since  $\text{ind}(\tilde{u}_i) = 1$ ,  $g(\tilde{u}_i) = 0$ , and  $\#\Gamma_0 = 1$ , the automatic transversality holds.

**9.2.4. Description of the differential.** Given two multisets  $\gamma' = \prod \gamma_i^{m'_i}$  and  $\gamma = \prod \gamma_i^{m_i}$ , we set  $\gamma/\gamma' = \prod \gamma_i^{m_i - m'_i}$  if  $m'_i \leq m_i$  for all  $i$ ; otherwise we set  $\gamma/\gamma' = 0$ .

We now prove the following lemma, which describes the differential  $\partial_0$  on  $ECC(V, \alpha) \otimes ECC(N, \alpha)$  in some detail:

**Lemma 9.2.3.** *The differential  $\partial_0$  is given by:*

$$(9.2.1) \quad \partial_0(\gamma \otimes \Gamma) = (\partial_V \gamma) \otimes \Gamma + (\gamma/e') \otimes h\Gamma + (\gamma/h') \otimes e\Gamma + \gamma \otimes (\partial_N \Gamma).$$

Here  $\gamma$  is an orbit set of  $V$ ; if  $h$  divides  $\Gamma$ , then  $h\Gamma$  is understood to be 0; and  $\partial_X$  is the differential on the subset  $X \subset M$ .

*Proof.* In view of Remark 9.1.2 and the blocking lemma (Lemma 5.7.1), the differential  $\partial_0$  does not count holomorphic curves which cross  $\mathbb{R} \times T_1$  or  $\mathbb{R} \times T_2$ . This still allows for the possibility of curves that are positively asymptotic to orbits of  $T_1$  or  $T_2$ ; such curves are contained in one of  $\mathbb{R} \times V$ ,  $\mathbb{R} \times T^2 \times [1, 2]$ , or  $\mathbb{R} \times N$ . Moreover, by Lemma 9.2.2, the only curves that intersect  $\mathbb{R} \times T^2 \times [1, 2]$  are branched covers of  $Z_{s,\theta}$ , provided we ignore the curves which are asymptotic to the orbits in the interior of  $T^2 \times [1, 2]$ .

We can now apply Proposition 5.4.2. There is a unique Morse-Bott building  $\tilde{u}_2$  from  $h'$  to  $e$  and a unique Morse-Bott building  $\tilde{u}_1$  from  $e'$  to  $h$ , modulo translations in the  $s$ -direction. Also, there are two cylinders from  $e'$  to  $h'$  and two cylinders from  $h$  to  $e$ ; these give  $\partial_0 e' = 0$  and  $\partial_0 h = 0$ . Branched covers of  $Z_{s,\theta}$  of degree  $> 1$  are eliminated since all Morse-Bott buildings in  $\mathbb{R} \times T^2 \times [1, 2]$  must be nice. Finally, all the Morse-Bott buildings in  $\mathbb{R} \times N$  (and similarly in  $\mathbb{R} \times V$ ) must be nice and the Morse-Bott family  $\mathcal{N} = \mathcal{N}_1$  can only be used at the negative end since no curve of  $\partial_0$  can cross  $T_1$ .  $\square$

**9.3. Proof of Theorem 9.1.3.** In this subsection we complete the proof of Theorem 9.1.3, i.e., we compute the  $E^1$ -term of the spectral sequence for  $\mathcal{F}$ . Recall that  $\partial_0$ , as described in Lemma 9.2.3, is the  $E^0$ -differential for the spectral sequence induced by  $\mathcal{F}$ , i.e., the differential of the associated graded complex of  $\mathcal{F}$  (identified, as a vector space, with  $ECC(V, \alpha) \otimes ECC(N, \alpha)$  itself).

9.3.1. *More filtrations.* We rewrite the differential  $\partial_0$  in a way which highlights the roles played by the orbits  $h$  and  $h'$ . This serves as a motivation for introducing the two filtrations  $\mathcal{G}$  and  $\mathcal{G}'$  which are used in the proof of Theorem 9.1.3.

By factoring out the terms  $h'$  and  $h$ , we can write the differentials  $\partial_V$  and  $\partial_N$  as:

$$\begin{cases} \partial_V \gamma = \partial_V^b \gamma \\ \partial_V(h' \gamma) = h' \partial_V^b \gamma + \partial'_V(h' \gamma) \end{cases} \quad \begin{cases} \partial_N \Gamma = \partial_N^b \Gamma + h \partial'_N \Gamma \\ \partial_N(h \Gamma) = h \partial_N^b \Gamma \end{cases}$$

where  $\gamma \in ECC^b(V)$ ,  $\Gamma \in ECC^b(N)$ ,  $\partial_V^b$  and  $\partial_N^b$  are the differentials for the chain complexes  $ECC^b(V)$  and  $ECC^b(N)$ , and the terms  $\partial'_V(h' \gamma)$  and  $\partial'_N \Gamma$  do not contain  $h'$ .

If we write  $C_{k,k'} = (h')^{k'} ECC^b(V, \alpha) \otimes h^k ECC^b(N, \alpha)$ , then

$$ECC(V, \alpha) \otimes ECC(N, \alpha) = C_{0,0} \oplus C_{0,1} \oplus C_{1,0} \oplus C_{1,1}.$$

We can organize all components of the differential  $\partial_0$  besides  $\partial_V^b \otimes 1$  and  $1 \otimes \partial_N^b$  by the following diagram:

$$(9.3.1) \quad \begin{array}{ccc} C_{0,1} & \xrightarrow{1 \otimes h \partial'_N + \cdot / e' \otimes h} & C_{1,1} \\ \partial'_V \otimes 1 + \cdot / h' \otimes e \downarrow & & \downarrow \partial'_V \otimes 1 + \cdot / h' \otimes e \\ C_{0,0} & \xrightarrow{1 \otimes h \partial'_N + \cdot / e' \otimes h} & C_{1,0} \end{array}$$

Inspired by the diagram, we introduce two filtrations of length 2 on  $ECC(V, \alpha) \otimes ECC(N, \alpha)$ :

- the *horizontal filtration*  $\mathcal{G}$ , which, on the generators  $\gamma \otimes \Gamma$ , gives the multiplicity of  $h$  in  $\Gamma$ ; and
- the *vertical filtration*  $\mathcal{G}'$  which, on the generators  $\gamma \otimes \Gamma$ , gives the multiplicity of  $h'$  in  $\gamma$ .

Since the filtrations are mutually orthogonal, each induces a filtration on the graded complex of the other. In fact, we will consider  $\mathcal{G}'$  on the graded complex of  $\mathcal{G}$ .

We end the subsection with an apology to the reader. We will use the spectral sequences associated to  $\mathcal{G}$  and  $\mathcal{G}'$  which, of course, are nothing but long exact sequences in homology. The use of spectral sequences is motivated by our wish to give a parallel treatment of the cases where we filter by the multiplicity of a hyperbolic orbit or by the multiplicity of an elliptic orbit. We could also combine the two filtrations into a single filtration of length three and study the corresponding spectral sequence. This route, although feasible, leads to some unpleasant computational complications.

9.3.2. *Completion of the proof of Theorem 9.1.3.* We consider the filtration  $\mathcal{G}$  on  $(ECC(V, \alpha) \otimes ECC(N, \alpha), \partial_0)$  which, on the generators  $\gamma \otimes \Gamma$ , gives the multiplicity of  $h$  in  $\Gamma$ . This means that  $\mathcal{G}(\gamma \otimes \Gamma)$  is either 0 or 1. Moreover, the differential  $\partial_0$  is  $\mathcal{G}$ -filtration nondecreasing since  $h$  cannot be at the positive end of a Morse-Bott building  $u$  in  $\mathbb{R} \times N$ .

The  $E^r$ -differential of the spectral sequence induced by  $\mathcal{G}$  will be denoted by  $\partial_{0r}$ . All pages inherit a grading  $E_k^r(\mathcal{G})$ ,  $k = 0, 1$ , from  $\mathcal{G}$ . Again we can identify the graded complex  $E^0(\mathcal{G})$  with the original complex  $ECC(V, \alpha) \otimes ECC(N, \alpha)$ , but only as vector spaces. With this identification, the differential  $\partial_{00}$  on  $E^0(\mathcal{G})$  can be written as:

$$(9.3.2) \quad \partial_{00}(\gamma \otimes h^k \Gamma) = (\partial_V \gamma) \otimes h^k \Gamma + \gamma/h' \otimes eh^k \Gamma + \gamma \otimes h^k (\partial_N^b \Gamma).$$

Here  $k = 0, 1$  and  $\Gamma$  does not have any  $h$  term.

Since the complex  $(E^0(\mathcal{G}), \partial_{00})$  is still not as simple as we would like, we introduce the filtration  $\mathcal{G}'$  on  $E^0(\mathcal{G})$  which, on the generators  $\gamma \otimes \Gamma$ , counts the multiplicity of  $h'$  in  $\gamma$ . The differential  $\partial_{00}$  is  $\mathcal{G}'$ -filtration nonincreasing. If we write the differential on  $E^r(\mathcal{G}')$  as  $\partial_{00r}$ , then  $\partial_{000}$  is given by:

$$\partial_{000}((h')^{k'} \gamma \otimes h^k \Gamma) = (h')^{k'} (\partial_V^b \gamma) \otimes h^k \Gamma + \gamma \otimes h^k (\partial_N^b \Gamma).$$

Here  $k' = 0, 1$ ,  $h'$  does not divide  $\gamma'$ , and  $h$  does not divide  $\Gamma$ . Observe that, once again, we may identify the original complex with its associated graded complex, of course only as vector spaces. Since the filtrations  $\mathcal{G}$  and  $\mathcal{G}'$  are orthogonal,  $E^r(\mathcal{G}')$  splits as the direct sum of two spectral sequences  $E_k^r(\mathcal{G}')$  for  $k = 0, 1$ , each of which is graded by  $\mathcal{G}'$ . Hence we have four groups  $E_{k,k'}^r(\mathcal{G}')$  which, at the  $r = 0$  level, correspond to the groups  $C_{k,k'}$  in Diagram (9.3.1).

Each of the  $C_{k,k'} = E_{k,k'}^0(\mathcal{G}')$  is now a product complex, and its homology is isomorphic to  $ECH^b(V) \otimes ECH^b(N)$ . Hence

$$\begin{aligned} E^1(\mathcal{G}') &= H_*(ECC(V, \alpha) \otimes ECC(N, \alpha), \partial_{000}) \\ &= \mathbb{F}[h', h] \otimes ECH^b(V) \otimes ECH^b(N). \end{aligned}$$

Here  $\mathbb{F}[h', h]$  denotes the polynomial ring generated by  $h'$  and  $h$ , where the generators are considered as Grassmann variables of odd degree. Since  $ECH^b(V) \simeq \mathbb{F}[e']$  by Corollary 8.0.5, we obtain:

$$E^1(\mathcal{G}') = \mathbb{F}[e', h', h] \otimes ECH^b(N).$$

The homology class of  $\gamma$  in  $H_1(V)$  defines a grading on  $ECC(V, \alpha)$ , which corresponds to the grading on  $E^1(\mathcal{F})$ . Since  $ECH(V, \alpha)$  is concentrated in the homology class  $0 \in H_1(V)$ , well-known properties of spectral sequences imply that  $E_p^1(\mathcal{F}) = 0$  for  $p > 0$ . This proves the  $p > 0$  part of Theorem 9.1.3.

Now we continue the computation of  $E_0^1(\mathcal{F})$ . The differential  $\partial_{001}$  is induced by the part of  $\partial_{00}$  which decreases the multiplicity of  $h'$  and is given by:

$$\begin{aligned} (e')^m \otimes h^k \Gamma &\mapsto 0, \\ (e')^m h' \otimes h^k \Gamma &\mapsto (e')^m \otimes h^k \Gamma + (e')^m \otimes e h^k \Gamma, \end{aligned}$$

since  $h' \mapsto \emptyset + e$ . In other words,  $(E^1(\mathcal{G}'), \partial_{001})$  is given by:

$$0 \rightarrow E_{k,1}^1(\mathcal{G}') \xrightarrow{\cdot/h' \otimes (1+e)} E_{k,0}^1(\mathcal{G}') \rightarrow 0$$

for  $k = 0, 1$ . A straightforward calculation gives  $E_{k,1}^2(\mathcal{G}') = 0$  for  $k = 0, 1$ , and

$$\begin{aligned} E^2(\mathcal{G}') &= E_{0,0}^2(\mathcal{G}') \oplus E_{1,0}^2(\mathcal{G}') \\ &\simeq \mathbb{F}[e', h] \otimes \lim_{j \rightarrow \infty} ECH_j^b(N) \\ &\simeq \mathbb{F}[e', h] \otimes ECH(N, \partial N). \end{aligned}$$

Since  $\mathcal{G}'$  has length 2, the spectral sequence for  $\mathcal{G}'$  converges at the  $E^2$ -term. Moreover,  $E_{k,0}^2(\mathcal{G}') \cong E_k^1(\mathcal{G})$ , where the isomorphism is induced by the inclusion  $E_{k,0}^0(\mathcal{G}') \subset E_k^1(\mathcal{G})$ ; this is due to the fact that  $E^2(\mathcal{G}')$  is concentrated in the lowest degree.

Returning to the spectral sequence for  $\mathcal{G}$ ,

$$(9.3.3) \quad E^1(\mathcal{G}) \simeq \mathbb{F}[e', h] \otimes ECH(N, \partial N).$$

Now, we compute that:

$$(9.3.4) \quad \partial_{01} : (e')^m \otimes \Gamma \mapsto (e')^{m-1} \otimes h\Gamma + (e')^m \otimes h\partial'_N \Gamma,$$

$$(9.3.5) \quad \partial_{01} : (e')^m \otimes h\Gamma \mapsto 0.$$

If we take into account the grading, then we can rewrite  $(E^1(\mathcal{G}), \partial_{01})$  as

$$E_{0,0}^1(\mathcal{G}) \xrightarrow{\cdot/e' \otimes h+1 \otimes h\partial'_N} E_{1,0}^1(\mathcal{G}),$$

where  $E_{0,0}^1(\mathcal{G}) = \mathbb{F}[e'] \otimes ECC(N, \partial N)$  and  $E_{1,0}^1(\mathcal{G}) = \mathbb{F}[e'] \otimes hECC(N, \partial N)$ .

**Lemma 9.3.1.** *The map  $\partial_{01} : E_{0,0}^1(\mathcal{G}) \rightarrow E_{1,0}^1(\mathcal{G})$  is surjective.*

*Proof.* We will define a right inverse  $s : E_{1,0}^1(\mathcal{G}) \rightarrow E_{0,0}^1(\mathcal{G})$  for the map  $\partial_{01}$  given by (9.3.4). First consider the map

$$\begin{aligned} s_0 : \mathbb{F}[e'] \otimes hECH^b(N) &\rightarrow \mathbb{F}[e'] \otimes ECH^b(N), \\ (e')^k \otimes h\Gamma &\mapsto (e')^k \sum_{i=1}^{\infty} (e')^i \otimes (\partial'_N)^{i-1} \Gamma. \end{aligned}$$

This map is well-defined because  $\partial'_N$  is nilpotent by action considerations. Hence  $s_0$  descends to the map  $s : E_{1,0}^1(\mathcal{G}) \rightarrow E_{0,0}^1(\mathcal{G})$ . One easily verifies that  $\partial_{01} \circ s = id$ .  $\square$

By Lemma 9.3.1,  $E_1^2(\mathcal{G}) = 0$  and

$$E^2(\mathcal{G}) = E_0^2(\mathcal{G}) = \ker(\partial_{01} : E_{0,0}^1(\mathcal{G}) \rightarrow E_{1,0}^1(\mathcal{G})).$$

An element of  $E_{0,0}^1(\mathcal{G}) = \mathbb{F}[e'] \otimes ECH(N, \partial N)$  has the form

$$(e')^n \otimes \Gamma_n + (e')^{n-1} \otimes \Gamma_{n-1} + \cdots + 1 \otimes \Gamma_0,$$

where  $\Gamma_i \in ECH(N, \partial N)$ , and it is easy to see that it is a  $\partial_{01}$ -cycle if and only if  $\Gamma_{i+1} = \partial'_N \Gamma_i$ ,  $i = 0, 1, \dots$ . Hence there is an identification  $ECH(N, \partial N) \simeq E_0^2(\mathcal{G})$  given by  $\Gamma \mapsto 1 \otimes \Gamma + f(\partial'_N \Gamma)$  and the spectral sequence for  $\mathcal{G}$  converges at the  $E^2$ -term. Finally,

$$E^1(\mathcal{F}) \simeq E^2(\mathcal{G}) \simeq ECH(N, \partial N),$$

since  $E^2(\mathcal{G})$  is concentrated in a single degree. This completes the proof of Theorem 9.1.3.

**9.4. Proof of Theorem 1.0.1(2).** Recall that  $\widehat{ECH}(M)$  is the homology of the mapping cone of the  $U$ -map  $ECC(M, \alpha) \rightarrow ECC(M, \alpha)$ . After substituting  $ECC(M, \alpha)$  by  $ECC(V, \alpha) \otimes ECC(N, \alpha)$ , we obtain the chain complex

$$C_U := (ECC(V, \alpha) \otimes ECC(N, \alpha)) \oplus (ECC(V, \alpha) \otimes ECC(N, \alpha)),$$

with differential  $\widehat{\partial} = \begin{pmatrix} \partial & 0 \\ U & \partial \end{pmatrix}$ . Here the elements of  $C_U$  are viewed as column matrices.

We define the filtration  $\widehat{\mathcal{F}}$  on  $C_U$  as follows:

$$\widehat{\mathcal{F}} \left( \begin{pmatrix} \sum_i \gamma_i \otimes \Gamma_i \\ \sum_j \gamma'_j \otimes \Gamma'_j \end{pmatrix} \right) = \max_{i,j} \{ \eta([\gamma_i]), \eta([\gamma'_j]) \}.$$

The filtration  $\widehat{\mathcal{F}}$  is analogous to the filtration  $\mathcal{F}$  which was used in Section 9.1. Let us write  $\widehat{\partial}_0 = \begin{pmatrix} \partial_0 & 0 \\ U_0 & \partial_0 \end{pmatrix}$  for the  $E^0$ -differential of  $\widehat{\mathcal{F}}$ . As usual, we identify  $E^0(\widehat{\mathcal{F}}) \simeq C_U$  as vector spaces.

**Claim 9.4.1.** *Let  $z$  be a generic point in the interior of  $\mathbb{R} \times T^2 \times [1, 2]$ . Then the  $U_0$ -map with respect to  $z$  is given by:*

$$U_0(\gamma \otimes \Gamma) = \gamma/e' \otimes e\Gamma.$$

*Proof.* If  $u(F)$  is a holomorphic curve which is counted in the  $U$ -map, then the filtration  $\widehat{\mathcal{F}}$  first restricts the slope of  $\pi_M(u(F)) \cap T_{2-\varepsilon}$  to  $\infty$ , as discussed in Lemma 9.1.1 and Remark 9.1.2. The computation of  $U_0$  then follows from the discussion of the finite energy foliation on  $\mathbb{R} \times T^2 \times [1, 2]$  in Section 9.2. Recall that  $\mathbb{R} \times \text{int}(T^2 \times [1, 2])$  is foliated by holomorphic cylinders, and there is a unique cylinder which passes through any given interior point  $z$ . Hence, after perturbing the Morse-Bott family of orbits on  $\partial(T^2 \times [1, 2])$ ,  $U_0$  is a count of a single holomorphic curve, namely a cylinder from  $e'$  to  $e$  which passes through  $z$ .  $\square$

Next consider the following filtration on  $ECC(V, \alpha) \otimes ECC(N, \alpha)$ :

$$\mathcal{E} \left( \sum_i \gamma_i \otimes \Gamma_i \right) = \max_i m(\gamma_i),$$

where  $m(\gamma)$  is the multiplicity of  $e'$  in  $\gamma$ . The  $U_0$ -map is strictly filtration-decreasing, i.e.,  $\mathcal{E}(U_0(\gamma \otimes \Gamma)) < \mathcal{E}(\gamma \otimes \Gamma)$ . This motivates the definition of the filtration  $\widehat{\mathcal{E}}$  on  $(E^0(\widehat{\mathcal{F}}), \partial_0)$ , where

$$\widehat{\mathcal{E}} \left( \sum_i \gamma_i \otimes \Gamma_i \right) = \max_{i,j} \{m(\gamma_i), m(\gamma'_j)\}.$$

Let  $\widehat{\partial}_{0r}$  be the  $E^r$ -differential of  $\widehat{\mathcal{E}}$ . Then  $(E^0(\widehat{\mathcal{E}}), \widehat{\partial}_{00})$  splits into two direct summands, each of the form  $(ECC(V) \otimes ECC(N), \partial_{00})$ . The differential  $\partial_{00}$  — which is not the same  $\partial_{00}$  defined in the previous subsection — is given by:

$$\partial_{00}(\gamma(e')^n \otimes \Gamma) = (\partial_V \gamma)(e')^n \otimes \Gamma + \gamma(e')^n \otimes \partial_N \Gamma + (\gamma/h')(e')^n \otimes e\Gamma,$$

where  $e'$  does not divide  $\gamma$ .

**Lemma 9.4.2.**

$$(9.4.1) \quad H_*(ECC(V) \otimes ECC(N), \partial_{00}) \simeq \mathbb{F}[e'] \otimes \widehat{ECH}(N, \partial N).$$

*Proof.* This calculation is similar to that of Section 9.3. In the present case we filter  $(ECC(V) \otimes ECC(N), \partial_{00})$  by the multiplicity of  $h'$  and apply Theorem 8.0.1, i.e.,  $ECH(\text{int}(V)) \simeq \mathbb{F}$  and is generated by the empty set.  $\square$

Now,  $E^1(\widehat{\mathcal{E}})$  is the direct sum of two copies of (9.4.1), and can be written as:

$$E^1(\widehat{\mathcal{E}}) = \mathbb{F}[e'] \otimes \left( \widehat{ECH}(N, \partial N) \oplus \widehat{ECH}(N, \partial N) \right).$$

The differential  $\widehat{\partial}_{01}$  maps:

$$(9.4.2) \quad (e')^n \otimes \begin{pmatrix} \Gamma \\ \Gamma' \end{pmatrix} \mapsto (e')^{n-1} \otimes \begin{pmatrix} h\Gamma \\ \Gamma + h\Gamma' \end{pmatrix},$$



for  $n = 0, 1, \dots$ . Here we set  $(e')^{-1} = 0$ . Observe that  $e\Gamma = \Gamma$  when  $\Gamma \in \widehat{ECH}(N, \partial N)$ .

Since the filtration  $\widehat{\mathcal{E}}$  induces a grading on  $E^r(\widehat{\mathcal{E}})$  which counts the multiplicity of  $e'$ , we can write  $E^r(\widehat{\mathcal{E}}) = \bigoplus_{n \geq 0} E_n^r(\widehat{\mathcal{E}})$ .

**Lemma 9.4.3.**

$$E_n^2(\widehat{\mathcal{E}}) \simeq \begin{cases} \widehat{ECH}(N, \partial N), & \text{if } n = 0; \\ 0, & \text{if } n > 0. \end{cases}$$

*Proof.* Let  $(e')^n \otimes \begin{pmatrix} \Gamma \\ \Gamma' \end{pmatrix} \in E_n^1(\widehat{\mathcal{E}})$ , where  $\Gamma, \Gamma' \in \widehat{ECH}(N, \partial N)$ . If  $n > 0$ , it is in  $\ker \widehat{\partial}_{01}$  if and only if  $\Gamma = h\Gamma'$ . On the other hand, such an element is the boundary of  $(e')^{n+1} \otimes \begin{pmatrix} \Gamma' \\ 0 \end{pmatrix}$ . Hence  $E_n^2(\widehat{\mathcal{E}}) = 0$  if  $n > 0$ .

When  $n = 0$ ,  $\ker \widehat{\partial}_{01}$  is all of  $E_0^1(\widehat{\mathcal{E}}) \cong \widehat{ECH}(N, \partial N) \oplus \widehat{ECH}(N, \partial N)$ . The lemma then follows by observing that  $E_0^1(\widehat{\mathcal{E}})$  is a direct sum of two subspaces  $\text{Im}(\widehat{\partial}_{01}) = \left\{ \begin{pmatrix} h\Gamma \\ \Gamma + h\Gamma' \end{pmatrix} \right\}$  and  $\widehat{ECH}(N, \partial N) \oplus \{0\}$ : Given  $\begin{pmatrix} a \\ b \end{pmatrix} \in E_0^1(\widehat{\mathcal{E}})$ , we can write it as  $\begin{pmatrix} h\Gamma \\ \Gamma + h\Gamma' \end{pmatrix} + \begin{pmatrix} \Gamma'' \\ 0 \end{pmatrix}$ , where  $\Gamma' = 0$ ,  $\Gamma = b$ , and  $\Gamma'' = hb + a$ . On the other hand, if  $\begin{pmatrix} h\Gamma \\ \Gamma + h\Gamma' \end{pmatrix} + \begin{pmatrix} \Gamma'' \\ 0 \end{pmatrix} = 0$ , then  $\Gamma = h\Gamma'$  and  $\Gamma'' = h(h\Gamma') = 0$ .  $\square$

The spectral sequence for  $\widehat{\mathcal{F}}$  thus converges to  $\widehat{ECH}(N, \partial N)$  at the  $E^1$ -term. This completes the proof of Theorem 1.0.1(2).

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